Q-Function

Alternative Representation of the Gaussian Q Function

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Abstract – Lately a new form of the Gaussian Q function has been used and it has been shown to be very useful in evaluating the probability of error performance for various modulation schemes over generalized fading channels. In this work, we will demonstrate this new form (definite integral) and we will try to give another easier approach to have such forms. The basic trick was to introduce the unit step function in the integral of the Q function and then replace it with its Fourier transform. This will produce a multi-dimensional integral that can be evaluated with the use of multivariate Gaussian Joint PDFs in matrix form. We will try to extend our approach to 2-dimentional alternative Gaussian Q-function Q(x,y,ρ), and focus on a specific case of it where x = y and $\rho = 0$.

I. Introduction

Q function is widely used in communication and this is because of its special mathematical representation and characteristics. It is defined as the complement of the CDF corresponding to a normalized Gaussian random variable X. The canonical representation of Q(x) is a semi finite integral of the PDF of the corresponding RV.

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

This representation suffers from 2 disadvantages. When using algorithmic techniques or numerical integral evaluation, it is better to have a truncated upper limit, and this happens in the case of pure AWGN channel. The other problem is the presence of the variable x in the lower limit of the integral, and this is a problem when x depends on other random parameters that require further statistical averaging over their probability distribution, and this is the case for fading channels.

$$\tilde{r}(t) = \sum_{l=1}^{Lp} \alpha_l e^{-j\theta_l} \tilde{s}(t - \tau_l) + \tilde{n}(t)$$

And the BER would be

$$P_b(E; \{\alpha_l\}) = Q\left(\sqrt{g \frac{2E_b}{N_0} \eta}\right)$$

where parameter η is a function of the set of fading amplitudes $\{\alpha_l\}$, g is a constant that differs per each modulation and $\frac{E_b}{N_0}$ is the received SNR per bit.

In the latter the argument of Q depends on the fading amplitudes of the received signals. Thus to evaluate the *average* BER we must average the BER (which is a Q function) over the fading amplitude distributions.

$$P_b(E) = \int_0^{\infty} Q\left(\sqrt{g \frac{2E_b}{N_0} \eta}\right) p_\eta(\eta)\, d\eta = \int_0^{\infty} \int_{g\frac{2E_b}{N_0}\eta}^{\infty} e^{-\frac{y^2}{2}} p_\eta(\eta)\, dy\, d\eta$$

And here comes the problem in which it is hard to evaluate the average over a random variable that appears in the lower limit of an integral. Therefore a lot research over several years was done to try to put the random variable inside the integral. Craig found an alternative form.

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\frac{x^2}{sin^2\theta}\right)} d\theta\ , x\ \geq 0$$

Using Craig's form it will be easier to evaluate the average BER for several modulation schemes under fading channels.

$$P_b(E) = \frac{1}{\pi} \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\frac{(g\frac{2E_b}{N_0}\eta)^2}{sin^2\theta}\right)} p_\eta(\eta) d\theta\ d\eta$$

Now having the random variable independent of the integral limits, we can change the order of the integrals and hereby ease its evaluation.

$$P_b(E) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-\frac{1}{2}\left(\frac{(g\frac{2E_b}{N_0}\eta)^2}{sin^2\theta}\right)} p_\eta(\eta)\, d\eta\, d\theta$$

This also extends to the 2-Dimensional Q – function where in several modulation schemes the SER rate is expressed in terms of

$$Q\ (x, y) = \int_x^{\infty}\int_y^{\infty} f_Z(z_{1,}z_2)\ dz$$

where

$$f_Z(z_{1,}z_2) = \frac{1}{(\sqrt{2\pi})^2\sqrt{|\det[Cz]|}} e^{-\frac{1}{2}z^T[C_Z]^{-1}z}$$

is the Gaussian joint PDF of the joint Gaussian RVs Z_1 and Z_2. Then additional averaging is needed in terms of the probability distributions of x and y and we will face the same problem as in the 1-Dimensional Q –function where the random variables we are averaging over are in the limits of the integral.

For example for the case of M-QAM where (M =2^k), the SER rate is expressed as

$$P_s(E;\gamma_S) = 4\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{2g_{QAM}\gamma_S}\right) - 4\left(1 - \frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{2g_{QAM}\gamma_S}\right)$$

where g_{QAM} = 3/[2(M-1)] and γ_S is the received SNR per symbol.

Thus the *average* SER would be the averaging of SER = $P_s(E;\gamma_S)$ over the distributions of γ_S.

$$P_s(E) = \int_0^{\infty} 4\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{2g_{QAM}\gamma_S}\right) - 4\left(1 - \frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{2g_{QAM}\gamma_S}\right) p_{\gamma_S}(\gamma_S)\, d\gamma_S$$

We are averaging over $\gamma_S$which is in the limits of the integral of a Q^2 - function.

To avoid this problem we would also need some kind of closed form expression of $Q(x, y; \rho)$ where we can evaluate

$$Q^2(x) = Q(x, x; \rho = 0)$$

where the RVs are inside the integral so we can also make a change in the order of the integral to integrate first the RVs.

Simon found a closed form expression for $Q(x, y; \rho)$

$$Q(x, y; \rho) = \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}-tan^{-1}\frac{y}{x}} \frac{\sqrt{1-\rho^2}}{1-\rho sin2\theta} e^{\left(-\frac{x^2}{2}\frac{1-\rho sin2\theta}{(1-\rho^2)sin^2\theta}\right)} d\theta + \frac{1}{2\pi}\int_{0}^{tan^{-1}\frac{y}{x}} \frac{\sqrt{1-\rho^2}}{1-\rho sin2\theta} e^{\left(-\frac{y^2}{2}\frac{1-\rho sin2\theta}{(1-\rho^2)sin^2\theta}\right)} d\theta,\ \ x, y \geq 0$$

Moreover, from the above equation he found a closed form expression for $Q^2(x)$

$$Q^2(x) = \frac{1}{\pi}\int_{0}^{\frac{\pi}{4}} e^{-\frac{1}{2}\left(\frac{x^2}{sin^2\theta}\right)} d\theta\ , x\ \geq 0$$

which can solve our problem and ease the evaluation of the BER.

Now in this paper we will try to show an alternate approach for the 1-Dimensional Craig's form of the Q- function as well as we will try to give an alternate form of the 2-Dimensional Q function whose integral is independent of the RVs in the case of $Q^2(x)$. Our approach will be based on using an easier way to reach the required results. The basic trick was to introduce the unit step function in the integral of the Q function and then replace it with its Fourier transform. This will produce a multi-dimensional integral that can be evaluated with the use of multivariate Gaussian Joint PDFs in matrix form. We will try to extend our approach to 2-dimentional alternative Gaussian Q-function $Q(x, y, \rho)$, and focus on a specific case of it where $x\ =\ y$ and $\rho = 0$.

II. Alternative Formulation of the Q Function

A. 1-Dimensional Gaussian Q-Function

The alternative representation of the Q function derived by Craig

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{1}{2}\left(\frac{x^2}{sin^2\theta}\right)}\, d\theta \,, x \ \geq 0$$

In the following we show how to derive these representations in a natural manner.

The 1-Dimensional Q- function is the probability that the real Gaussian variable $Y \sim \mathcal{N}(0,1)$ satisfies

$$Q\ (x) = P(Y > x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{\frac{-y^2}{2}}\, dy$$

Using the unit step function, this can be written as

$$Q\ (x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}}\, u\ (y-x) dy$$

$u\ (y-x)$ upon using its Fourier transform is

$$u(y-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(y-x)(j\omega+\ \beta)}}{j\omega +\ \beta}\, d\omega$$

And therefore this will result to

$$Q\ (x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(y-x)(j\omega+\ \beta)}}{j\omega +\ \beta} d\omega dy$$

where we complete the square with respect to y

$$Q\ (x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-x(j\omega+\ \beta)}}{j\omega +\ \beta}\frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(j\omega+\ \beta)^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2y(j\omega+\ \beta)+\ (j\omega+\ \beta)^2)}\, dyd\omega$$

and by realizing the inner integral sums out to unity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\ (j\omega+\ \beta))^2}\, dy = 1$$

since referring to the Eq (3.462 - 4) in the textbook [3]

$$\int_{-\infty}^{\infty} x^n\, e^{-(x-\alpha)^2} dx = (2i)^{-n}\sqrt{\pi} H_n(i\alpha)$$

Selecting $n\ =\ 0$ and $x = \frac{y}{\sqrt{2}}$ and $\alpha = \frac{(j\omega+\ \beta)}{\sqrt{2}}$, we will have:

$$\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-(\frac{y}{\sqrt{2}}-\frac{(j\omega+\ \beta)}{\sqrt{2}})^2} dy = \sqrt{\pi}\, H_0\left(i\frac{(j\omega +\ \beta)}{\sqrt{2}}\right)$$

But $H_0\left(i\frac{(j\omega+\beta)}{\sqrt{2}}\right) = 1$, thus

$$\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-(\frac{y}{\sqrt{2}}-\frac{(j\omega+\beta)}{\sqrt{2}})^2} dy = \sqrt{\pi}$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\ (j\omega+\beta))^2} dy = 1$$

Then,

$$Q\ (x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-x(j\omega+\beta)+\frac{1}{2}(j\omega+\beta)^2}}{j\omega+\beta} d\omega$$

Now introduce the change of variable $\omega = -\beta\cot\theta, then\ d\omega = \beta(1+cot^2\theta)d\theta$

$$Q\ (x) = \frac{1}{2\pi}\int_{0}^{\pi} \frac{e^{-x(-\beta jcot\theta+\beta)+\frac{1}{2}(-\beta jcot\theta+\beta)^2}}{-j\beta cot\theta+\beta}\beta(1+cot^2\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{(-x\beta jcot\theta-x\beta)+\frac{1}{2}(-\beta^2 cot\theta^2+\beta^2-2j\beta^2cot\theta)}\ (1+jcot\theta)d\theta$$

Knowing $x \geq 0$, set $\beta = x\ \geq 0$. Then

$$Q\ (x) = \frac{1}{2\pi}\int_{0}^{\pi} e^{(-x^2 jcot\theta-x^2)+\frac{1}{2}(-x^2cot\theta^2+x^2-2jx^2cot\theta)}\ (1+jcot\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{-x^2+\frac{1}{2}(-x^2cot\theta^2+x^2)}\ (1+jcot\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{-\frac{1}{2}x^2(1+cot\theta^2)}\ (1+jcot\theta)d\theta$$

The imaginary part is odd and hence integrates to zero while the even part can be simplified to

$$Q\ (x) = \frac{1}{2\pi}\int_{0}^{\pi} e^{-\frac{1}{2}\left(\frac{x^2}{sin^2\theta}\right)} d\theta$$

$sin^2\theta$ is even in the interval $[0\ ,\pi]$; therefore,

$$Q(x) = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} e^{-\frac{1}{2}(\frac{x^2}{sin^2\theta})}\ d\theta$$

which is the form derived by Craig.

B. 2-Dimensional Gaussian Q-Function

Simon derived an alternative form for the 2- Dimensional Q function

$$Q(x,y;\rho) = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}-tan^{-1}\frac{y}{x}} \frac{\sqrt{1-\rho^2}}{1-\rho sin2\theta} e^{\left(-\frac{x^2}{2}\frac{1-\rho sin2\theta}{(1-\rho^2)sin^2\theta}\right)} d\theta + \frac{1}{2\pi}\int_0^{tan^{-1}\frac{y}{x}} \frac{\sqrt{1-\rho^2}}{1-\rho sin2\theta} e^{\left(-\frac{y^2}{2}\frac{1-\rho sin2\theta}{(1-\rho^2)sin^2\theta}\right)} d\theta,\ \ x,y \geq 0$$

We notice that the variables x and y are inside just one integral and no more double integral.

Now we will extend our derivation to the 2-Dimensional Q function, we will try to use the method we used before to remove the variables (x, y) from the limits of the double integral and try to achieve a single integral instead of a double one.

The 2-Dimensional Q- function is the probability that the real joint Gaussian variables Z_1 and Z_2 having the joint PDF

$$f_Z(z_{1,}z_2) = \frac{1}{(\sqrt{2\pi})^2\sqrt{|\det[Cz]|}} e^{-\frac{1}{2}z^T[C_Z]^{-1}Z}$$

$$where\ [C_Z]_{i,j} = cov[z_i, z_j] = \rho_{z_i,z_j}\sigma_{z_i}\sigma_{z_j}\ and\ Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

satisfies

$$Q\ (x,y;\ \rho) = P(Z_1 > x\ and\ Z_2 > y) = \int_y^\infty \int_x^\infty f_Z(z_{1,}z_2)\, dz$$

Using the unit step function, this can be written as

$$Q\ (x,y)\ = \int_x^\infty \int_y^\infty f_Z(z_{1,}z_2)\ dz = \int_{-\infty}^\infty \int_{-\infty}^\infty f_Z(z_{1,}z_2)\, u(z1-x)u(z2-y)\ dz$$

$u(z1-x)$ and $u(z2-y)$ upon using their Fourier transform would be

$$u\ (z_1 - x)\ = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{(z_1-x)(j\omega_1+\ \beta_1)}}{j\omega_1 +\ \beta_1} d\omega_1$$

and

$$u\ (z_2 - y) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{(z_2-y)(j\omega_2+\ \beta_2)}}{j\omega_2 +\ \beta_2} d\omega_2$$

Consider:

$$u(z1-x)\,u(z2-y)$$

$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{(z_1-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}d\omega_1\int_{-\infty}^{\infty}\frac{e^{(z_2-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_2$$

$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(z_1-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(z_2-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2$$

$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}e^{(z_1)(j\omega_1+\beta_1)}e^{(z_2)(j\omega_2+\beta_2)}\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2$$

$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}e^{[j\omega_1+\beta_1\,,j\omega_2+\beta_2]\;Z}\;\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2$$

$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}e^{b^T Z}\;\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2$$

where

$$b^T=[\,j\omega_1+\beta_1\quad j\omega_2+\beta_2]$$

Therefore the Q function can be written as

$$Q\,(x,y)$$
$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\left(\sqrt{2\pi}\right)^2\sqrt{|\det[Cz]|}}e^{-\frac{1}{2}Z^T[C_Z]^{-1}Z}\;e^{b^T Z}\;\iint_{-\infty}^{\infty}\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}\;d\omega_1 d\omega_2 dz$$

$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\frac{1}{\left(\sqrt{2\pi}\right)^2\sqrt{|\det[Cz]|}}e^{-\frac{1}{2}Z^T[C_Z]^{-1}Z+b^T Z}\;\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2 dz$$

Rearranging into squared form

$$Q\,(x,y)$$
$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty}\frac{1}{\left(\sqrt{2\pi}\right)^2\sqrt{|\det[Cz]|}}e^{-\frac{1}{2}(Z-C_Z b)^T[C_Z]^{-1}(Z-C_Z b)+\frac{1}{2}b^T[C_Z]b}\;\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2 dz$$

Now we try to form a Gaussian PDF inside the integral

$$Q\,(x,y)$$
$$=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{1}{\left(\sqrt{2\pi}\right)^2\sqrt{|\det[Cz]|}}e^{-\frac{1}{2}(Z-C_Z b)^T[C_Z]^{-1}(Z-C_Z b)}\;dz\;\iint_{-\infty}^{\infty}e^{\frac{1}{2}b^T[C_Z]b}\;\frac{e^{(-x)(j\omega_1+\beta_1)}}{j\omega_1+\beta_1}\frac{e^{(-y)(j\omega_2+\beta_2)}}{j\omega_2+\beta_2}d\omega_1 d\omega_2$$

so the outer integral would sum out to unity

$$Q\ (x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} e^{\frac{1}{2}b^T[C_Z]b}\ \frac{e^{(-x)(j\omega_1+\ \beta_1)}}{j\omega_1 +\ \beta_1}\frac{e^{(-y)(j\omega_2+\ \beta_2)}}{j\omega_2 +\ \beta_2}d\omega_1 d\omega_2$$

Expanding the constant matrix, we will have

$$e^{\frac{1}{2}b^T[C_Z]b} = e^{\frac{(j\omega_1+\ \beta_1)^2\ +\ 2\rho(j\omega_1+\ \beta_1)(j\omega_2+\ \beta_2)+(\omega_2+\ \beta_2)^2}{2}}$$

And therefore

$$Q\ (x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)(j\omega_2 +\ \beta_2)}e^{\frac{(j\omega_1+\ \beta_1)^2\ +\ 2\rho(j\omega_1+\ \beta_1)(j\omega_2+\ \beta_2)+(\omega_2+\ \beta_2)^2}{2}}e^{(-x)(j\omega_1+\ \beta_1)}e^{(-y)(j\omega_2+\ \beta_2)}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}\int_{-\infty}^{\infty}\frac{e^{(-x)(j\omega_1+\ \beta_1)}}{(j\omega_1 +\ \beta_1)}e^{\frac{(j\omega_1+\ \beta_1)^2\ +\ 2\rho(j\omega_1+\ \beta_1)(j\omega_2+\ \beta_2)}{2}}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\frac{(j\omega_1+\ \beta_1)^2}{2}+\ [\rho(j\omega_2+\ \beta_2)-x](j\omega_1+\ \beta_1)}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}$$
$$* \int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\frac{(j\omega_1+\ \beta_1)^2}{2}+\ [\rho(j\omega_2+\ \beta_2)-x](j\omega_1+\ \beta_1)+\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\left[\frac{(j\omega_1+\ \beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\ \beta_2)-x}{\sqrt{2}}\right]^2}e^{-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\left[\frac{(j\omega_1+\ \beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\ \beta_2)-x}{\sqrt{2}}\right]^2}e^{-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}e^{-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\left[\frac{(j\omega_1+\ \beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\ \beta_2)-x}{\sqrt{2}}\right]^2}d\omega_1 d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}e^{-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\left[\frac{(j\omega_1+\ \beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\ \beta_2)-x}{\sqrt{2}}\right]^2}d\omega_1 d\omega_2$$

Trying to solve the integral:

$$Q(x,y) = \int_{-\infty}^{\infty}\frac{1}{(j\omega_2 +\ \beta_2)}e^{(-y)(j\omega_2+\ \beta_2)}e^{\frac{(j\omega_2+\ \beta_2)^2}{2}}e^{-\frac{[\rho(j\omega_2+\ \beta_2)-x]^2}{2}}\int_{-\infty}^{\infty}\frac{1}{(j\omega_1 +\ \beta_1)}e^{\left[\frac{(j\omega_1+\ \beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\ \beta_2)-x}{\sqrt{2}}\right]^2}d\omega_1 d\omega_2$$

Upon completing the squares of the outer integral, we will have

$$= \int_{-\infty}^{\infty} \frac{1}{(j\omega_2 + \beta_2)} e^{(-y)(j\omega_2+\beta_2)+\frac{(j\omega_2+\beta_2)^2}{2}-\frac{(\rho(j\omega_2+\beta_2)-x)^2}{2}} \int_{-\infty}^{\infty} \frac{1}{(j\omega_1 + \beta_1)} e^{\left[\frac{(j\omega_1+\beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\beta_2)-x}{\sqrt{2}}\right]^2} d\omega_1 d\omega_2$$

$$= e^{-\frac{(\rho x-y)^2}{2(1-\rho^2)}-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{(j\omega_2 + \beta_2)} e^{\frac{1-\rho^2}{2}\left[(j\omega_2+\beta_2)+\left(\frac{\rho x-y}{1-\rho^2}\right)\right]^2} \int_{-\infty}^{\infty} \frac{1}{(j\omega_1 + \beta_1)} e^{\left[\frac{(j\omega_1+\beta_1)}{\sqrt{2}}+\frac{\rho(j\omega_2+\beta_2)-x}{\sqrt{2}}\right]^2} d\omega_1 d\omega_2$$

Let $\omega_1 = -\beta_1 \cot\theta_1, then\ d\omega_1 = \beta_1(1 + cot^2\theta_1)d\theta_1$ and $\omega_2 = -\beta_2 \cot\theta_2,\ d\omega_2 = \beta_2(1 + cot^2\theta_2)d\theta_2$

$$= e^{-\frac{(\rho x-y)^2}{2(1-\rho^2)}-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\beta_2(1 - j cot\theta_2)} e^{\frac{1-\rho^2}{2}\left[\beta_2(1-j cot\theta_2)+\left(\frac{\rho x-y}{1-\rho^2}\right)\right]^2} \int_{-\infty}^{\infty} \frac{1}{\beta_1(1 - j cot\theta_1)} e^{\left[\frac{\beta_1(1-j cot\theta_1)}{\sqrt{2}}+\frac{\rho\beta_2(1-j cot\theta_2)-x}{\sqrt{2}}\right]^2}$$
$$* \ \beta_1(1 + cot^2\theta_1)\beta_2(1 + cot^2\theta_2)d\theta_1 d\theta_2$$
$$= e^{-\frac{(\rho x-y)^2}{2(1-\rho^2)}-\frac{x^2}{2}} \int_0^{\pi} (1$$
$$+ j\cot\theta_2) e^{\frac{1-\rho^2}{2}\left[\beta_2(1-j\cot\theta_2)+\left(\frac{\rho x-y}{1-\rho^2}\right)\right]^2} \int_0^{\pi} (1 + j\cot\theta_1) e^{\left[\frac{\beta_1(1-j\cot\theta_1)}{\sqrt{2}}+\frac{\rho\beta_2(1-j\cot\theta_2)-x}{\sqrt{2}}\right]^2} d\theta_1 d\theta_2$$

III. Conclusion

In this paper, we have shown an alternative form of the Gaussian Q-function and a new formulation to achieve this. Our approach was based on removing the variables (x & y) from the limits of the integrals by using the trick of unit step function than replacing the latter by its Fourier transform and hereby ending up with multi-dimensional integral that can be computed with the use of the multivariate Gaussian Joint PDFs and some change of variable. Since the limits of the integral are independent of the variables (x & y), this will make it easier to average BER and compute PER and thus deal with performance evaluation of several modulation techniques over fading channels. We applied this on 1-Dimentional Q-function and we ended up with the same alternate form derived by Craig, then we extended it to the 2-Dimensional Q-function where we ended up with an integral shape whose limits are independent of (x & y). Then we moved to a special case of the 2-Dimentional Q- function $Q^2(x)$ which is used in evaluating the PER of QAM modulation techniques.

IV. References

[1] M.K Simon and M.S. Alouini, “A Unified Approach to the Performance Analysis of Digital Communications over Generalized Fading Channels”, *Proc. IEEE*, vol. 86, no. 9, Sep. 1998, pp. 1860 – 1877.

[2] T.Y. Al-Naffouri, B. Hassibi, “On the Distribution of Indefinite Quadratic Forms in Gaussian Random Variables” *under preparation.*

[3] Gradshteyn and Ryzhik, *Table of Integrals, Series and Products.* Alan Jeffrey and Daniel Zwillinger(eds), 5th edition, 1996

[4] M.K. Simon and M.S. Alouini, *Digital Communication over Fading Channels,* 2nd edition*.* Wiley, N.J. 2002