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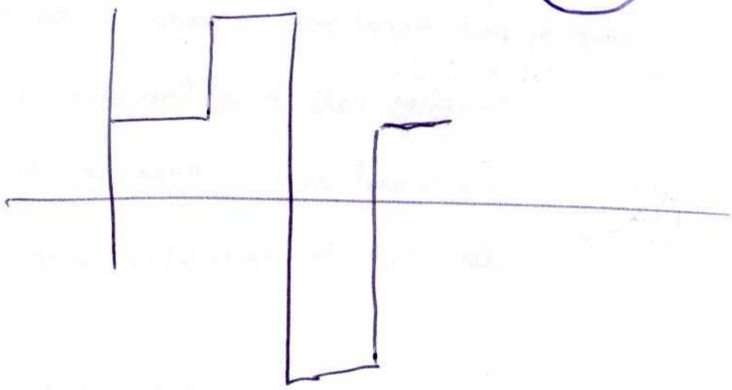
Pulse-amplitude-modulated (PAM) Signals
(See Proakis 169–171)

~~$s_m(t)$~~
PAM as we know it



More generally, in PAM, the waveforms are given by

$$0 \leq t \leq T$$

$$\cancel{x}_m(t) = A_m \, g(t) \cos 2\pi f_c t$$

$g(t)$: instead of a rect.

$\cos 2\pi f_c t$: want to move signal passband (around f_c)

$$= \mathrm{Re}\left\{ A_m \, g(t) \, e^{j2\pi f_c t} \right\}$$

$A_m \quad m = 1, \ldots, M \qquad$ M possible amplitudes

If $M = 2^k$, each symbol represents k bits

One basis function $\varphi(t) = g(t) \cos 2\pi f_c t$

To make basis orthonormal, divide by energy

$$\int \varphi^2(t) = \frac{1}{2}\int_0^T g^2(t) + \frac{1}{2}\int_0^T g^2(t) \cos 2\pi (2 f_c t)$$

$\simeq 0$ (Accurate when $f_c \gg$ BW of $g(t)$)

$$= \frac{\mathcal{E}_g}{2}$$

(we don't want to have analysis depend on carrier freq.)

$$\Rightarrow \varphi(t) = \frac{\varphi'(t)\sqrt{2}}{\sqrt{\mathcal{E}_g}}$$

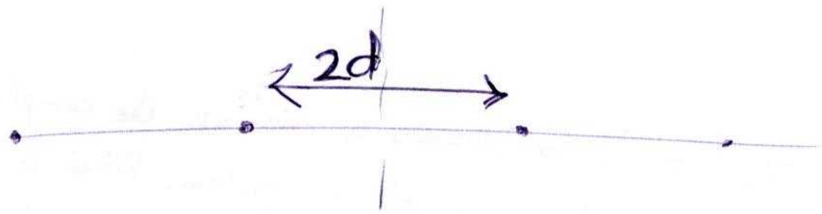
$$\Rightarrow \boxed{\varphi(t) = \sqrt{\frac{2}{\mathcal{E}_g}}\, g(t) \cos 2\pi f_c t}$$

$$\boxed{x_m = A_m \sqrt{\frac{\mathcal{E}_g}{2}}} \qquad \text{as } x_m(t) = A_m \sqrt{\frac{\mathcal{E}_g}{2}}\, \varphi(t)$$

Now we know how we want to choose the A_m's

choose **them so that** they have zero mean

M=2



- In general, we have

$$A_m = (2m-1-M)d \qquad m = 1, 2, \dots, M$$

(assuming M is even) because then avg. of constellation is zero.

- What is A_m if M is odd?
- If $M = 2^k$, each symbol represents k bits

~~Heat~~ R = bit rate $\qquad T_b = 1/R$ bit interval

symbol rate $= R/k$

How do we distribute the bits ?

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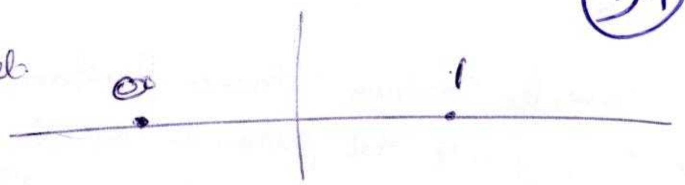
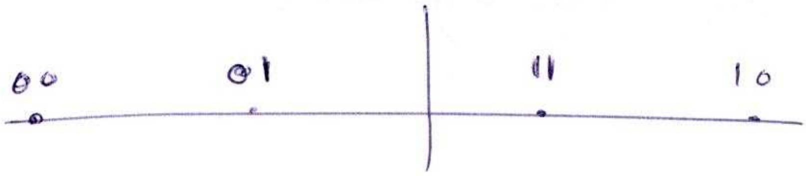
Use Gray coding
(only one difference bet. adjacent symbols)





Gray coding : adjacent symbols ~~have only~~ are different in one bit only

$\Rightarrow$ Since most errors happen ~~bet.~~ adjacent errors with high probability

$\Rightarrow$ one symbol error results in one bit error

Avg energy $\quad \mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m \quad$ (equally probable symbols)

$$= \frac{1}{M} \sum_{m=1}^{M} \frac{\mathcal{E}_g}{2} \cdot A_m^2$$

$$= \frac{\mathcal{E}_g}{2M} \sum_{m=1}^{M} (2m-1-M)d^2$$

$$= \frac{d^2 \mathcal{E}_g}{6} (M^2-1)$$

d_{min} = distance bet. any two adjacent points

$$= |x_{m+1} - x_m|$$

$$= \sqrt{\frac{\mathcal{E}_g}{2}} \, |A_{m+1} - A_m|$$

$$= d\sqrt{2\mathcal{E}_g}$$

Distance bet. any ~~two~~ constellation points

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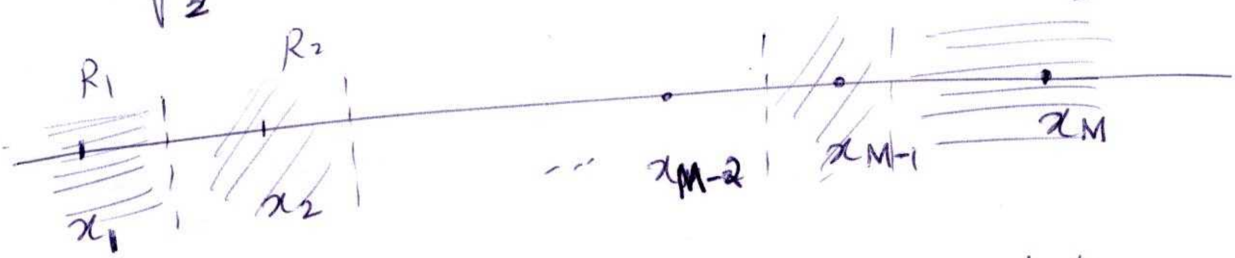
$$d_{mn} = |x_m - x_n|$$
$$= \sqrt{\frac{\mathcal{E}_g}{2}}\, d\, |A_m - A_n|$$
$$= \sqrt{\frac{\mathcal{E}_g}{2}}\, d\, |m-n|$$

important for union bound for error prob.

Prob. of error

Received signal

$$r = x_m + n \qquad \mathcal{N}\left(0, \frac{N_0}{2}\right)$$
$$= \sqrt{\frac{\mathcal{E}_g}{2}} A_m + n$$



Decision regions are decided by the Euclidean distance

For each inner point ($M-2$ of them)

$$P_e = P(|n| > \text{half dist. bet. adj pts})$$
$$= 2\, P\left(n > \frac{d_{min}}{2}\right)$$
$$= 2\, Q\left(\frac{d_{min}/2}{\sqrt{N_0/2}}\right) = 2\, Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$
$$= 2\, Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right)$$

External points (2 of them)

$$P_e = P\left(n > \frac{d_{min}}{2}\right)$$

$$= Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right)$$

$$\Rightarrow \quad P_e = \frac{1}{M}\left(2\,Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right) + 2(M-2)\,Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right)\right)$$

$$= \frac{2(M-1)}{M}\,Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right)$$

Express result in terms of avg. energy of constellation

$$\mathcal{E}_{avg} = \frac{d^2 \mathcal{E}_g}{6}\,(M^2-1)$$

$$\Rightarrow \quad P_e = \frac{2(M-1)}{M}\,Q\left(\sqrt{\frac{6\,\mathcal{E}_{avg}}{(M^2-1)N_0}}\right)$$

No use average energy per bit instead

M symbols transmit k bits

$\mathcal{E}_{avg}$ $\qquad$ $\mathcal{E}_{b\,avg}$

$$\Rightarrow \quad \mathcal{E}_{b\,avg} = \frac{\mathcal{E}_{avg}}{k} = \frac{\mathcal{E}_{avg}}{\log_2 M}$$

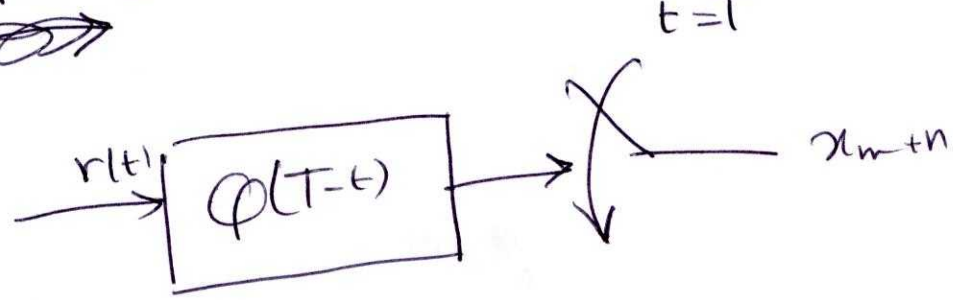
$$\Rightarrow \quad P_e = \frac{2(M-1)}{M}\,Q\left(\sqrt{\frac{6\log_2 M\ \mathcal{E}_{b\,av}}{(M^2-1)\,N_0}}\right)$$

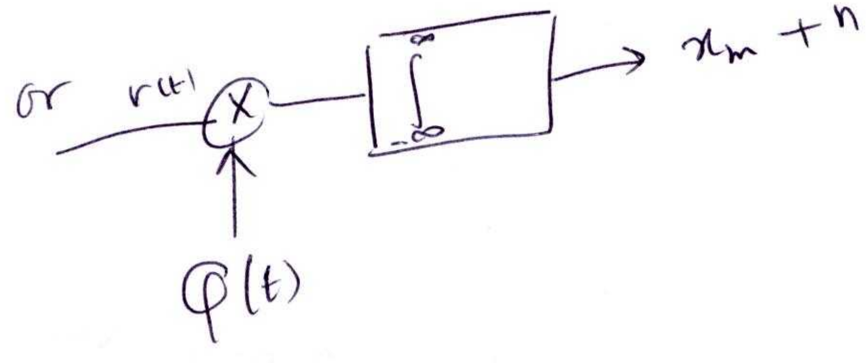
How do we demodulate?

$$x_m(t) = A_m \sqrt{\frac{\mathcal{E}_g}{2}}\,\varphi(t)$$

$$\varphi(t) = \sqrt{\frac{2}{\mathcal{E}_g}}\, g(t) \cos 2\pi f_c t$$

$$r(t) = x_m(t) + n(t)$$





We need to construct a $\varphi(t)$ for each freq.!

Is there a way around it?

Downconvert to an intermediate freq. & then demodulate

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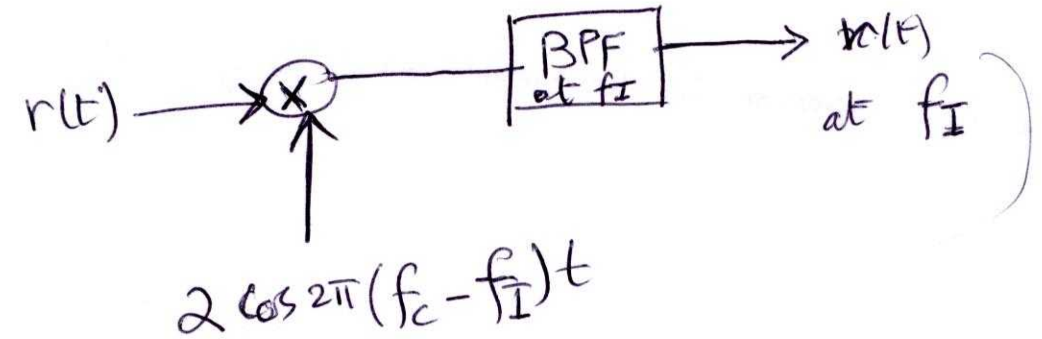
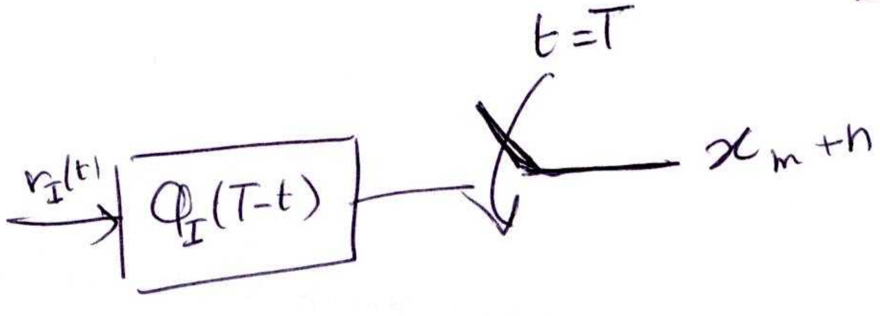


i.e.

$$r_I(t) = \left(A_m \sqrt{\frac{\mathcal{E}_g}{2}}\right) \underbrace{\left(\sqrt{\frac{2}{\mathcal{E}_g}}\, g(t) \cos 2\pi f_I t\right)}_{\varphi_I(t)} + n(t)$$



BW requirements of PAM :

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PAM waveform

$x_m(t) = A_m g(t) \cos 2\pi f_c t$

DSB-SC signal

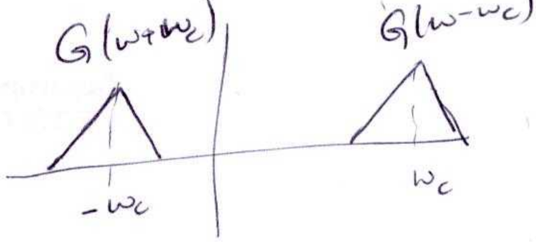
(Double side band)

$BW_{x_m(t)} = 2\, BW_{g(t)}$

$\Rightarrow$ BW efficiency = 50%

How to reduce BW requirement ?

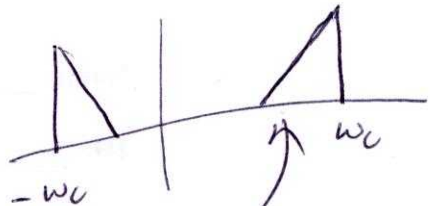
Two ways



(1) Use SSB modulation

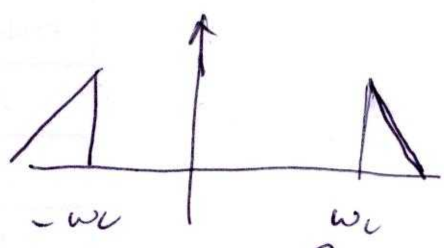
$x_m(t) = A_m g(t) \cos w_c t + A_m \hat{g}(t) \sin w_c t$

Hilber Transform



Lower side band

$x_m(t) = A_m g(t) \cos w_c t - A_m \hat{g}(t) \sin w_c t$



USB: Upper side band

$BW_{required} = BW_{g(t)}$

(2) Use baseband Transmission

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$x_m(t) = A_m g(t)$

$BW_{x_m(t)} = BW_{g(t)}$

Disadv. is that ~~we~~ can only use it at a particular frequency band $[0, BW_{g(t)}]$.

BW efficiency = 100%

(3) Use quadrature amplitude modulation

$\varphi_{QAM}(t) = g_1(t) \cos w_c t + g_2(t) \sin w_c t$

$BW_{g_1} = BW_{g_2} = B$ $\qquad BW_{QAM} = 2B$

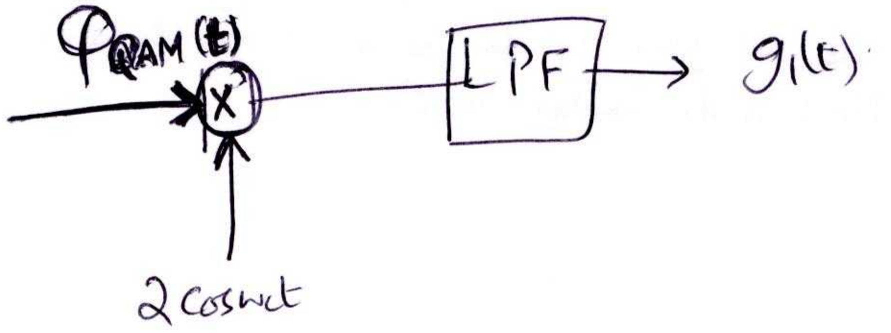
send two signals of $BW = B$ using a $BW = 2B$

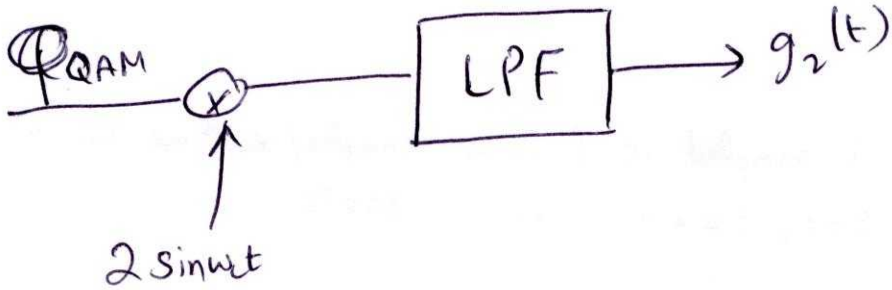
$\Rightarrow$ BW efficiency = 100%

~~In our digital modulation case, we set~~

~~$x_m(t) = A_{m_1} g(t) \cos w_c t + A_m$~~

How do we demodulate this signal





In our digital modulation case, ~~we set~~ send two levels in one signal, i.e.

$$\varphi_m(t) = A_{mc}\cos 2\pi f_c t - A_{ms}\sin 2\pi f_c t$$

a

Phase-modulated signals:

(Proakis 171 - 173)

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PAM $\longrightarrow$ Amplitude modulation

Phase modulated signal $\longrightarrow$ Phase modulation

$$x_m(t) = \mathrm{Re}\left[g(t)\, e^{j2\pi\frac{(m-1)}{M}}\, e^{j2\pi f_c t}\right] \quad m = 1, 2, \dots, M$$

$0 \leq t \leq T$

M possible phases

$$\Rightarrow x_m(t) = g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right]$$

Obtaining the spanning basis

$$x_m(t) = g(t)\cos\frac{2\pi}{M}(m-1)\cos 2\pi f_c t - g(t)\sin\left(\frac{2\pi}{M}\right)(m-1)\sin 2\pi f_c t$$

$$\varphi_1'(t) = g(t)\cos 2\pi f_c t$$

$$\Rightarrow \varphi_1(t) = +\sqrt{\frac{2}{\mathcal{E}_g}}\, g(t)\cos 2\pi f_c t$$

$$\varphi_2'(t) = -g(t)\sin 2\pi f_c t$$

$$\varphi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}}\, g(t)\sin 2\pi f_c t$$

$M = 2^3 = 8$

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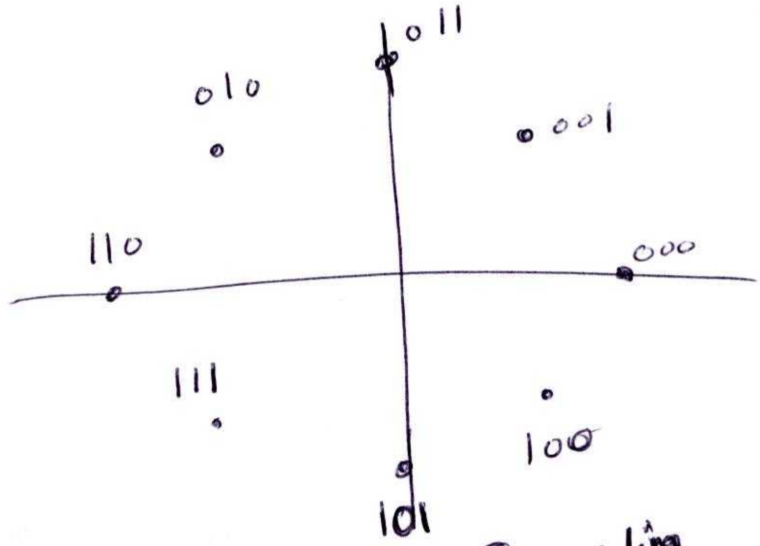


Gray coding

Avg. energy

$$\|x_m\|^2 = \frac{\mathcal{E}_g}{2} \cos^2 \frac{2\pi}{M}(m-1) + \frac{\mathcal{E}_g}{2} \sin^2 \frac{2\pi}{M}(m-1$$

$$= \mathcal{E}_g$$

($\Rightarrow$ what does this say a/b the detector?)

$\Rightarrow$ Avg energy $\mathcal{E}_{av} = \mathcal{E}_g$

Euclidean distance

$\cos(+)\cos(-)$

$$d_{mn} = \|x_m - x_n\|$$

$$= \sqrt{\frac{\mathcal{E}_g}{2}} \left\| \begin{bmatrix} \cos \frac{2\pi}{M}(m-1) - \cos \frac{2\pi}{M}(n-1) \\ \sin \frac{2\pi}{M}(m-1) - \sin \frac{2\pi}{M}(n-1) \end{bmatrix} \right\|$$

$\sin(+)\cos(-)$

$$= \sqrt{\mathcal{E}_g} \left[1 - \cos \frac{2\pi}{M}(m-n) \right]^{1/2}$$

Note that

$$\int_0^T \varphi_1(t)\varphi_2(t)\,dt = \frac{-1}{\mathcal{E}_g}\int_0^T g^2(t)\sin 2\pi(2f_c)t\,dt$$

$$\simeq 0$$

$$\Rightarrow \quad x_m(t) = \sqrt{\frac{\mathcal{E}_g}{2}}\cos\left(\frac{2\pi}{M}(m-1)\right)\varphi_1(t) +$$

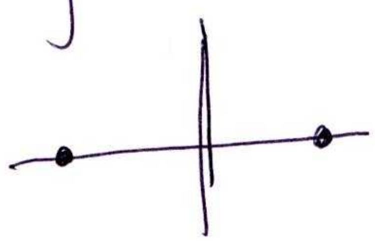
$$\sqrt{\frac{\mathcal{E}_g}{2}}\sin\left(\frac{2\pi}{M}(m-1)\right)\varphi_2(t)$$

$$\Rightarrow \quad x_m = \begin{bmatrix} \sqrt{\frac{\mathcal{E}_g}{2}}\cos\left(\frac{2\pi}{M}(m-1)\right) \\ \sqrt{\frac{\mathcal{E}_g}{2}}\sin\left(\frac{2\pi}{M}(m-1)\right) \end{bmatrix}$$

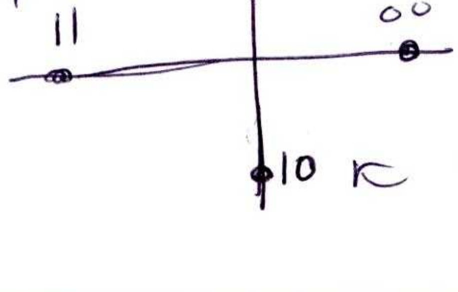
divide 2π into $\frac{2\pi}{M}$ parts & put one signal there

$M = 2^1$

$$x_1 = \begin{bmatrix} \sqrt{\frac{\mathcal{E}_g}{2}} \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -\sqrt{\frac{\mathcal{E}_g}{2}} \\ \end{bmatrix}$$



$M = 2^2 = 4$



What is the prob. of error?

Gray coding

$$d_{min} = \sqrt{\mathcal{E}_g\left(1 - \cos\frac{2\pi}{M}\right)}$$

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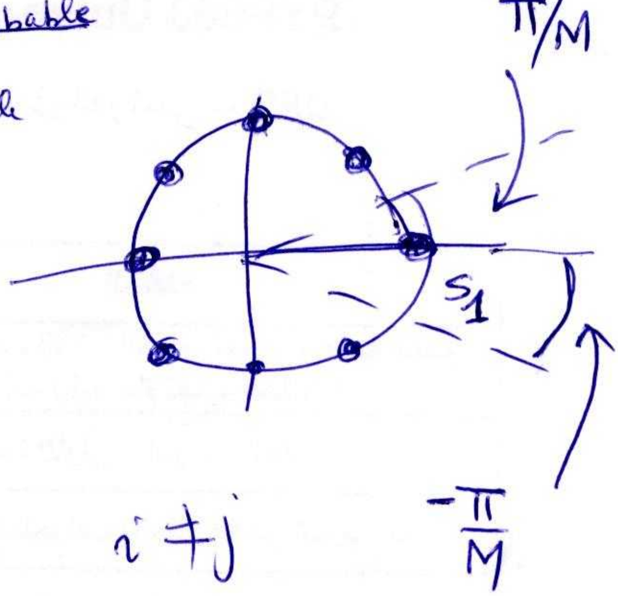
Probability of error ~~266~~ Proakis 266 - 269 49

~~All signals are equiprobable~~

All signals are equiprobable



AWGN

$\Rightarrow$ choose s_i if

$$\|r - s_i\|^2 < \|r - s_j\|^2 \quad i \neq j$$

Signals are of equal energy $\Rightarrow$

choose s_i if

$$r \cdot s_i > r \cdot s_j$$

$$\Leftrightarrow \cancel{|r|}\cancel{|s_i|}\cos\theta_i > \cancel{|r|}\cancel{|s_j|}\cos\theta_j$$

$$\Leftrightarrow \cos\theta_i > \theta_j$$

$$\Leftrightarrow \theta_i < \theta_j$$

angle bet. r &

Since all signals are equiprobable

$$P_e = P_e|s_1 \text{ transmitted}$$

$$= 1 - P_c|s_1$$

$$= 1 - P\left\{-\frac{\pi}{M} \leq \theta_1 \leq \frac{\pi}{M}\right\}$$

So we need to find the prob. that the angle of r is in $[-\frac{\pi}{M}, \frac{\pi}{M}]$ given that s_1 is transmitted.

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$$r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\mathcal{E}_g}{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

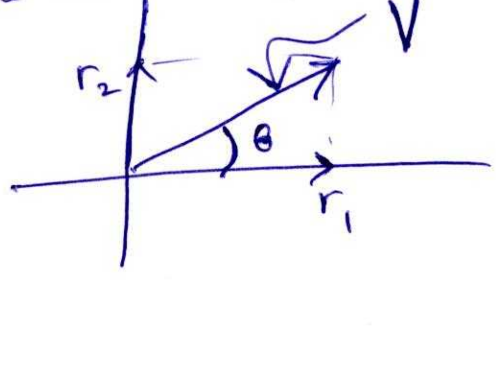
r_1 & r_2 are independent Gaussian r.v.'s

$r_1 \sim N\left(\sqrt{\frac{\mathcal{E}_g}{2}}, \frac{N_0}{2}\right)$

$r_2 \sim N\left(0, \frac{N_0}{2}\right)$

$$\Rightarrow p(r_1, r_2) = p(r_1)p(r_2) = \frac{1}{\left(\sqrt{2\pi\frac{N_0}{2}}\right)^2} e^{-\frac{\left(r_1-\sqrt{\frac{\mathcal{E}_g}{2}}\right)^2}{N_0}} e^{-\frac{r_2^2}{N_0}}$$

Want to find the distribution of θ of r



$$\overbrace{r_1}^{g_1^{-1}(v,\theta)} = V\cos\theta$$

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$$\underbrace{r_2}_{g_2^{-1}(v,\theta)} = V\sin\theta$$

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}}{\partial v} & \frac{\partial g_2^{-1}}{\partial \cancel{v}} \\ \frac{\partial g_1^{-1}}{\partial \theta} & \frac{\partial g_2^{-1}}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -V\sin\theta & V\cos\theta \end{vmatrix} = V\cos^2 + V\sin^2 = V$$

$$p(v,\theta) = p(V\cos\theta, V\sin\theta)\,|J|$$

$$= \frac{1}{\left(\sqrt{2\pi\frac{N_o}{2}}\right)^2} e^{-\frac{\left(V\cos\theta - \sqrt{\frac{\mathcal{E}_g}{2}}\right)^2}{N_o}} e^{-\frac{V^2}{N_o}\sin^2\theta}$$

$$= \frac{V}{2\pi\frac{N_o}{2}} \exp\left(-\frac{V^2 + \frac{\mathcal{E}_g}{2} - \sqrt{2\mathcal{E}_g}\,V\cos\theta}{N_o}\right)$$

$$P_\Theta = \int p(v,\theta)\,dv = \int \ldots \longrightarrow \frac{1}{2\pi} e^{-}$$

Can not integrate in closed form

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} P_\Theta(\theta)\,d\theta$$

So we need to go from rectangular to polar coordinates

$$V = \sqrt{r_1^2 + r_2^2}$$

$$\theta = \tan^{-1}\left(\frac{r_2}{r_1}\right)$$

1. From pdf of $p(r_1, r_2)$, we obtain pdf of $p(V, \theta)$.

2. From $p(V, \theta)$, we obtain $P_\theta(\theta)$. How?

3. Then $P_e = 1 - P_c$

$= ?$

$$= 1 - \int_{-\frac{\pi}{M}}^{\pi/M} P_\theta(\theta)\, d\theta$$