We would like to transform waveform signaling into the discrete signaling we are used to. (5)

So we digress to talk about Vector spaces and Inner product spaces.

Vector space

A vector space is similar to the space of real vectors $\mathbb{R}^3$ with vector addition & scalar multiplication

$$v_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad v_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$v_1 + v_2 = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

Scalar multiplication

$$\alpha v_1 = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{bmatrix}.$$

⑥

Linear independence:

The vectors $x_1, x_2, \cdots, x_N$ of a vector space X

are linearly independent if

$$\sum c_i x_i = 0 \Rightarrow c_1 = c_2 = \cdots = c_N = 0$$

in other words, we can not express one vector

in terms of the other vectors.

Ex: Show that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ are lin indep.

Inner product space

Given two vectors x & y in a space X,

we can define the inner product of these

two vectors $\langle x, y \rangle$. The inner product satisfies

(it is a real number)

① $\langle x, y \rangle \in \mathbb{R}$

Example: The dot product that we know:

$a \cdot b = a_1 b_1 + a_2 b_2$

② $\langle x, y \rangle = \langle y, x \rangle$

③ $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

④ $\langle x, x\rangle = 0 \iff x = 0$ (the zero vector) ⑦

⑤ $\langle x, y\rangle = 0 \Rightarrow$ x & y are orthogonal

⑥ we call $\langle x, x\rangle$ the norm of x and denote it by $\|x\|^2$.

A basis of a vector space X is any set of ~~vector~~ linearly independent vectors that span X, i.e.

~~for any $x \in X$~~ Let $\{S_1, S_2, \ldots, S_N\}$ be a basis for X. Then $\forall x \in X$, $\exists\, c_1, c_2, \ldots, c_N$ such that

$$x = \sum c_i S_i$$

e.g. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ is a basis for $\mathbb{R}^2$ (orthogonal basis)

Similarly, $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ is a basis but it is not orthogonal.

(8)

- The basis is orthogonal if

$$\langle S_i, S_j \rangle = 0$$

- The basis is orthonormal if

$$\langle S_i, S_i \rangle = \|S_i\|^2 = 1$$

In a similar way, we can show that

$$\langle v_1, v_2 \rangle = \sum_{i=1}^{N} \alpha_i \beta_i$$

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So we don't worry about the S's any more. We just carry the α_i's & β_i's around.

What is the significance of having an orthonormal basis?

(16)

Let $\{s_1, s_2, \dots, s_N\}$ be an orthonormal basis.

Let v_1 & v_2 be two vectors. Then, we can write

$$v_1 = \sum \alpha_i s_i$$

$$v_2 = \sum \beta_i s_i$$

What is $\langle v_1, s_j \rangle$?

$$\langle v_1, s_j \rangle = \alpha_j$$

Now, what is $\|v_1\|^2$?

$$\|v_1\|^2 = \langle v_1, v_1 \rangle$$

$$= \left\langle \sum \alpha_i s_i, \sum \alpha_j s_j \right\rangle$$

$$= \sum_j \left\langle \sum \alpha_i s_i, \alpha_j s_j \right\rangle$$

$$= \sum_i \sum_j \langle \alpha_i s_i, \alpha_j s_j \rangle$$

$$= \sum_i \sum_j \alpha_i \alpha_j \langle s_i, s_j \rangle \qquad \langle s_i, s_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$= \sum_i \alpha_i^2$$

Signal space concept

Consider the space of all functions $f(t)$ such that $\int_0^T f^2(t)\,dt < \infty$

This is a vector space.

We define the inner product in this space as

$$\langle f(t), g(t) \rangle = \int_0^T f(t)\, g(t)\, dt$$

Check that this inner product satisfies the various properties we talked about.

~~Example Consider the~~

Example Consider the subspace of all periodic functions of period T. What is an orthogonal basis for this space?

An orthogonal basis is the set of sines & cosines

$\cos \frac{2\pi n t}{T}$ & $\sin \frac{2\pi n}{T} t$

Now, given a set of waveforms

$x_0(t), x_1(t), \ldots, x_{M-1}(t)$,

we can represent these waveforms using an orthonormal basis

i.e. we write

$$x_i(t) = \sum_{n=1}^{N} x_n^i \varphi_n(t)$$

The basis $\{\varphi_1(t), \ldots, \varphi_N(t)\}$ is orthonormal. This means that

$$\langle \varphi_i(t), \varphi_i(t) \rangle = \int_0^T \varphi_i^2(t) = 1$$

$$\langle \varphi_i(t), \varphi_j(t) \rangle = \int_0^T \varphi_i(t) \varphi_j(t) dt = 0$$

So, the $x_i(t)$'s map to the vector $\begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_N^i \end{bmatrix}$

~~Now we can carry the dot product~~

Now, we can calculate the energy of a waveform by calculating the energy of the corresponding vector

$$\int x^2(t)\,dt = \int \left(\sum_n x_n \varphi_n(t), \sum_m x_m \varphi_m(t) \right) dt$$

$$= \int \sum_n x_n^2 \varphi_n^2(t)\,dt \quad \text{(by orthogonality)}$$

$$= \sum_n x_n^2 \underbrace{\int \varphi_n^2(t)\,dt}_{1}$$

$$= \sum_n x_n^2$$

$\boxed{\text{Parseval's relation}}$

So energy of $x(t)$ is equal to energy of $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

In a similar manner, we can find the dot product between two waveforms

$x_i(t)$ maps into $x^i = \begin{bmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_N^i \end{bmatrix}$,

then $\langle x_i(t), x_j(t) \rangle = \int x_i(t)\, x_j(t)\, dt = x^{i\,T} x^j$

Once we have the vector x, we can define average energy of constellation, minimum distance, ..., etc.

Distance between two waveforms

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$$\|x_i(t) - x_j(t)\|^2 =$$

$$\langle x_i(t) - x_j(t),\ x_i(t) - x_j(t) \rangle$$

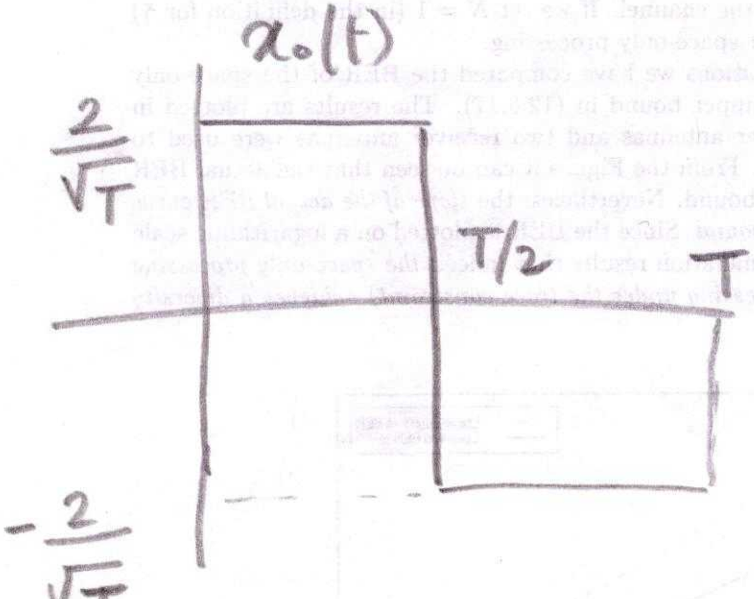
$$= \langle x_i(t), x_i(t) \rangle + \langle x_j(t), x_j(t) \rangle - 2 \langle x_i(t), x_j(t) \rangle$$

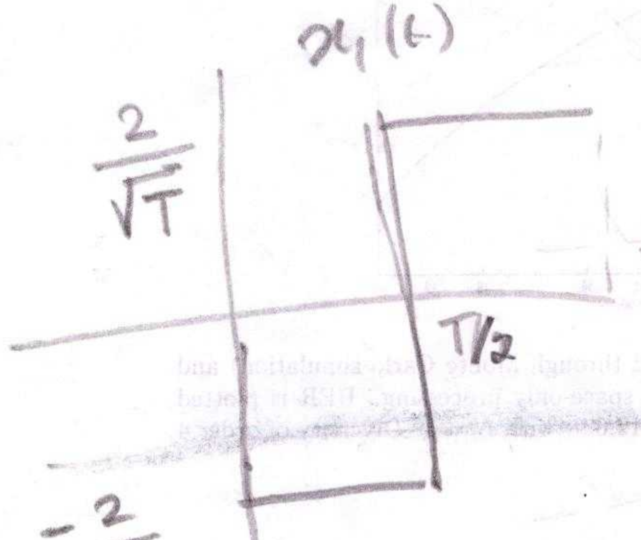
$$= \|x^i\|^2 + \|x^j\|^2 - 2\, x^{iT} x^j$$

$$= \|x^i - x^j\|^2$$

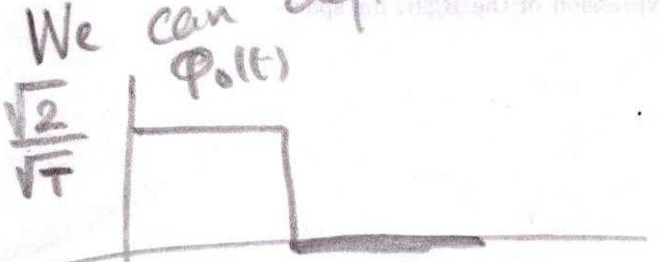
Euclidean distance

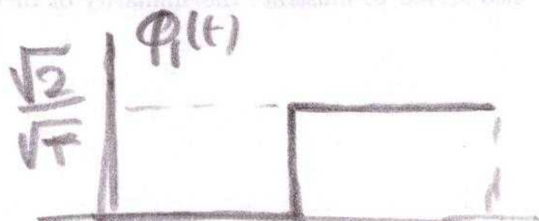
Ex Find an orthogonal basis for the two waveforms





We can define the following basis

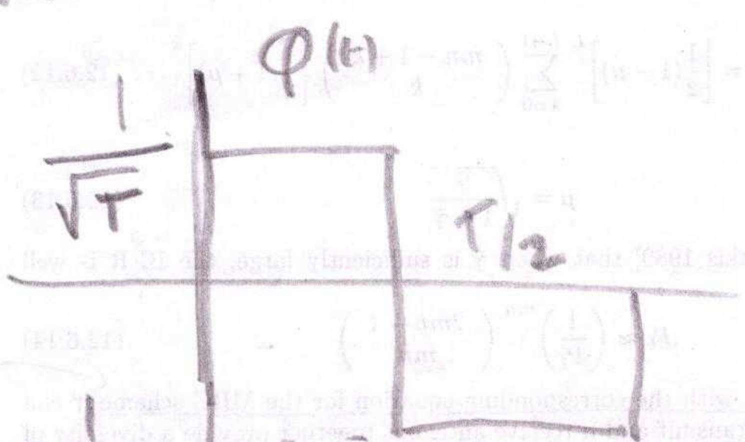




We can do better. (18)

- Note that if we rotate constellation, then two points become along the same line
- We don't need two basis vectors

One vector is enough



$$\int \varphi^2(t)\,dt = 1$$

$$x_0(t) = 2\varphi(t)$$

$$x_1(t) = -2\varphi(t)$$

In general We would like to have the least # of φ's

The modulators & demodulators will be much easier to build.

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Note that

(1) $\varphi_0(t)$ & $\varphi_1(t)$ are orthogonal

because $\varphi_0(t)\varphi_1(t) = 0 \Rightarrow$

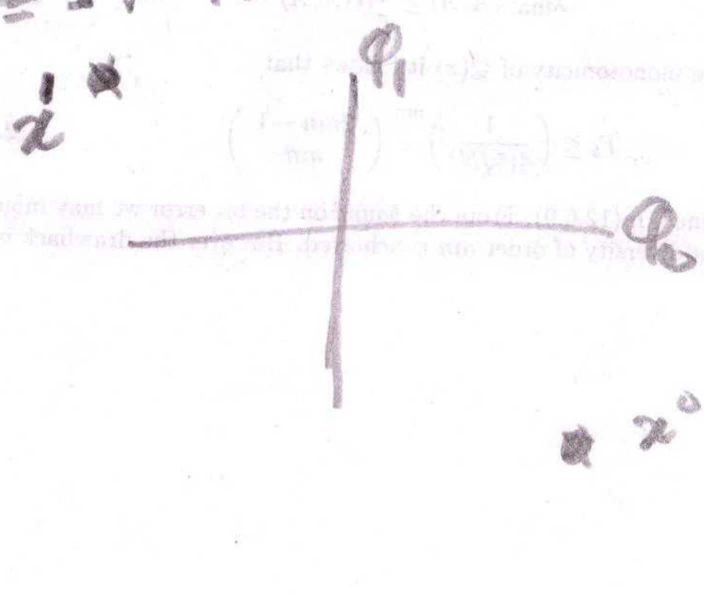
$$\int \varphi_0(t)\,\varphi_1(t)\,dt = 0$$

(2) $\int \varphi_0^2(t)\,dt = \int \varphi_1^2(t)\,dt = 1$

$\Rightarrow \{\varphi_0, \varphi_1\}$ is an orthonormal basis.

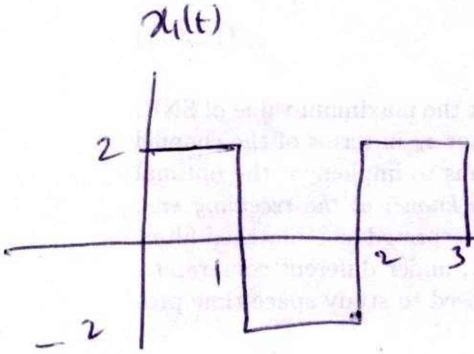
$$x_0(t) = \sqrt{2}\,\varphi_0(t) - \sqrt{2}\,\varphi_1(t) \qquad x^0 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

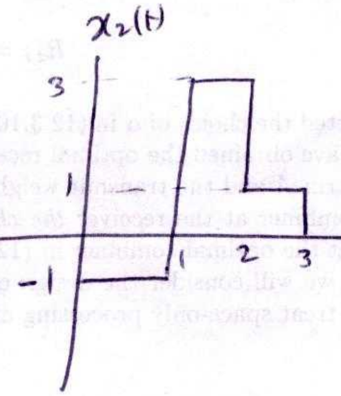
$$x_1(t) = -\sqrt{2}\,\varphi_0(t) + \sqrt{2}\,\varphi_1(t) \qquad x^1 = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

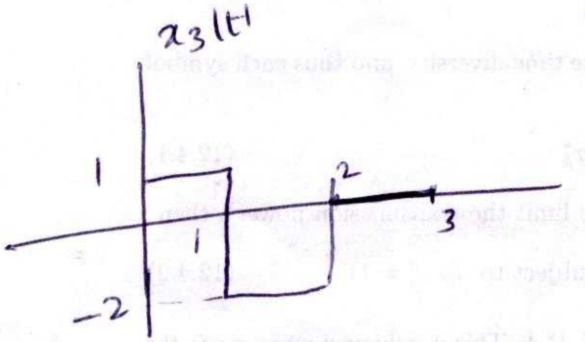


EX 2

Find a basis for the following waveforms 20







We can not find the $\phi_i(t)$'s by inspection

~~We need~~ a

We need a more general method for finding the orthonormal basis because 21

1. We would like to have the least # of $\varphi_n(t)$'s
2. Finding the $\varphi_n(t)$'s might not always be intuitive

To do this, we employ the Gram Shmidt Procedure.