Union Bound on the Prob. of Error for ML detection

Assume equi-probable signals

$$P_e = \frac{1}{M} \sum P_{e|m}$$

$$= \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p(r|s_m)\, dr$$

For AWGN channels

$$P_{e|m} = \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p(r|s_m)\, dr$$

$$= \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} p_n(r - s_m)\, dr$$

$$= \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \sum_{\substack{1 \le m' \le M \\ m' \ne m}} \int_{D_{m'}} e^{-\frac{\|r - s_m\|^2}{N_0}}\, dr$$

Simple to evaluate when decision regions are regular enough such that integral can be evaluated in closed form

We employ alternatives to union bound for two reasons.

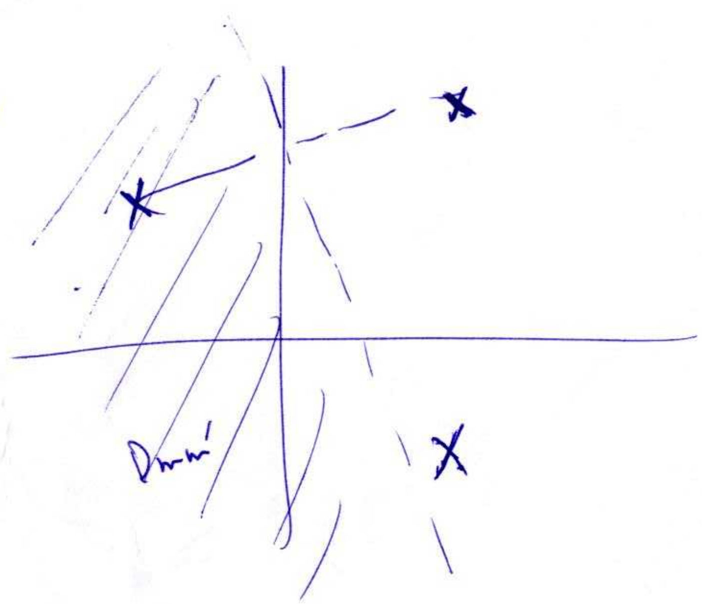
(1) When the integration regions are not regular enough

(2) When we want to express P_e in a unified way

$$P_e \le \sum Q\left(\sqrt{\frac{d}{N_0}}\right)$$

This is needed because we would like to compare various modulation technique.

~~Decision regions for ML detection~~ is



$D_{m'} = \{ r : p(r|s_{m'}) > p(r|s_k) \quad 1 \leq k \leq M,\ k \neq m' \}$

Define

$D_{mm'} = \{ p(r|s_{m'}) > p(r|s_m) \}$

$D_{m'} \subseteq D_{mm'}$

So

$$\int_{D_{m'}} p(r|s_m) \leq \int_{D_{mm'}} p(r|s_m)\, dr$$

error prob. of a binary equiprobable sys. with sig's s_m & $s_{m'}$

Pairwise error prob.

$$P_{m \to m'} = \int_{D_{mm'}} p(r|s_m)\, dr$$

$$P_{e|m} \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int_{D_{mm'}} p(r|s_m)\, dr$$

$$= \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} P_{m \to m'}$$

$$\Rightarrow P_e = \frac{1}{M} \sum_{m=1}^{M} P_{e|m}$$

$$\leq \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} P_{m \to m'}$$

For AWGN ch., we have

$$P_{m \to m'} = Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}} \right)$$

So union bound is given by

$$P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}} \right)$$

$$\Rightarrow P_e \leq \frac{1}{2M} \sum_{m=1}^{M} \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} e^{-\frac{d^2_{mm'}}{4N_0}}$$

as $Q(x) \leq \frac{1}{2} e^{-x^2/2}$.

Union bound in terms of min distance:

~~Def~~

Let $d_{min} = \min_{\substack{1 \leq m, m' \leq M \\ m \neq m'}} \|s_m - s_{m'}\|$

Since $Q(x)$ is a decreasing func, we have

$$Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}}\right) \leq Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$$

So $$\boxed{P_e \leq (M-1)\, Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)}$$

allows us to compare diff. mod. schemes.