Optimal detection for a General vector channel

Math model for AWGN ch. is

$$r = s_m + n$$

This is what we expect to get when project $r(t)$ on $\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)$.

Now r is a vector in the $\mathbb{R}^N$ space. How do we construct our decision regions to minimize the prob. of error.

When $n(t)$ is Gaussian, n will be Gaussian & so will be r.

<u>Consider the general case</u>:

$$s_m \longrightarrow \boxed{\begin{array}{c}\text{Channel}\\ p(r|s_m)\end{array}} \longrightarrow r$$

signal vectors ~~$\{s_m, 1 \leq m \leq M\}$~~ s_m $\{s_m \; 1 \leq m \leq N\}$ are selected from according to a priori probabilities P_m & transmitted over channel.

<u>Received Signal</u>

received sig. r depends statistically on Tx vector through cond. prob. density function

$$p(r|s_m)$$

<u>Receiver</u>

Observes r & based on this decides which message was transmitted

$$r \longrightarrow \boxed{g(r)} \longrightarrow \hat{m} \in \{1, 2, \ldots, M\}$$

$g(r) = \hat{m}$ receiver decides that $\hat{m}$ was transmitted

Prob. that decision is correct ~~is the probab~~ probability that $\hat{m}$ is the Tx message given that r is received

$$P[\text{correct decision} \mid r] = P[\hat{m} \text{ is sent} \mid r]$$

$$P[\text{correct decision}] = \int P[\text{correct} \mid r]\, p(r)\, dr$$
$$= \int P[\hat{m} \text{ sent} \mid r]\, p(r)\, dr$$

<u>Opt. detector</u>

Detector that minimizes prob. of error

Detector that max. prob. of correct decision

$$P[\text{correct decision}] = \int P[\hat{m} \text{ sent} \mid r]\, \underbrace{p(r)}_{\geq 0}\, dr$$

maximized if for each r $P[\hat{m} \text{ sent} \mid r]$ is maximized.

$$\hat{m} = g_{opt}(r)$$
$$= \arg\max_{1 \leq m \leq M} P[m \mid r]$$

Check all $P[m \mid r]$ $1 \leq m \leq N$ & select the largest.

or

$$\hat{m} = \arg\max_{1 \leq m \leq M} P[s_m \mid r]$$

MAP & ML Receivers

Decision rule

$$\boxed{\hat{m} = \arg\max_{1 \le m \le M} P[s_m | r]} \quad \star$$

is the max a posteriori probability rule (MAP)

Can be equivalently written as

$$\hat{m} = \arg\max_{1 \le m \le M} \frac{P_m \, p(r|s_m)}{\underbrace{p(r)}_{\text{indep. of } m \text{ for all } m}}$$

$$\Rightarrow \boxed{\hat{m} = \arg\max_{1 \le m \le M} P_m \, p(r|s_m)}$$ ← Easier to use than $\star$

because

P_m i are directly available

$p(r|s_m)$

~~MAP Recei~~

If the messages are equiprobable

$$P_m = \frac{1}{M}$$

optimal decision is

$$\boxed{\hat{m} = \arg\max_{1 \le m \le M} p(r|s_m)}$$ maximum likelihood Rx

ML receiver not optimal detector unless messages are equiprobable.

Popular since exact probabilities a/b messages difficult to get (e.g. info comes from different sources)

Decision Regions

Detector partitions space R^N into M regions $D_1, D_2, \dots, D_M$ s.t.

if $r \in D_m$, then $\hat{m} = g(r) = m$

D_m : is the set of channel outputs r that are mapped into msg m by detector

For MAP

$$D_m = \{ r \in \mathbb{R}^N : P[m|r] > P[m'|r] \quad \forall\, 1 \leq m' \leq M \quad m' \neq m \}$$

What happens if two messages are such that $P[m|r] = P[m'|r]$ for a given r

In that case, assign r to one of the 2 decision regions.

This will not affect prob. of error.

Error Probability

An error occurs if when s_m is transmitted, r is not in D_m

$\Rightarrow$ symbol error prob. with decision regions $\{D_m, 1 \leq m \leq M\}$ is given by

$$P_e = \sum_{m=1}^{M} P_m \, P[r \notin D_m | s_m \text{ sent}]$$

$$= \sum_{m=1}^{M} P_m P_{e|m}$$

$P_{e|m}$ = error prob. when msg m is tx

$$= \int_{D_m^c} p(r|s_m)\,dr$$

$$= \sum_{\substack{m'=1 \\ m' \neq m}}^{M} \int_{D_{m'}} p(r|s_m)\,dr$$