Proceedings of the International Conference on Computer and Communication Engineering 2008 May 13-15, 2008 Kuala Lumpur, Malaysia

Erasures Recoverability Analysis for SPC Product Codes

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Abstract

Single Parity Check (SPC) product codes are simple yet powerful codes that are used to correct errors and/or recover erasures. The focus of this paper is to evaluate the performance of such codes under erasure scenarios and to develop a tight upper bound for the post-decoding erasure rate. Closed form exact expressions are derived for up to seven erasures. The derived expressions are verified using exhaustive search. A bound is used to account for eight erasures and above. The developed expressions improve the evaluation of the recoverability of SPC product codes without the need for simulation or exhaustive search.

I. INTRODUCTION

Product codes were introduced by Elias in 1954. They represent the first method capable of achieving error-free coding with a nonzero code rate [1]. They are powerful codes that can be used to correct errors or recover erasures. Product codes are widely used in data-storage, optical and wireless applications. Product codes are useful in certain wireless network systems where packet information with a header is transmitted over a noisy channel like ATM cell wireless transmission [2].

Single-parity-check (SPC) product codes are the simplest form of two-dimensional product codes. A single parity bit is added at the end of every row and every column as depicted in Figure 1. The two-dimensional encoding scheme allows many patterns of lost cells to be recovered. This in turn results in significant reduction in the post-decoding cell loss rate.

SPC product codes have a minimum distance of four and hence using iterative decoding, all erasures of order three or less can be recovered. The code can recover many erasure patterns beyond those with the number of erasures determined by the minimum distance. Judging the erasure recovery performance of a product code based on its minimum distance is pessimistic. The work in [2] considered developing exact form for the post-decoding cell loss recovery for up to 5 errors. In [3-5] a procedure is proposed to perform recoverability analysis for the SPC product codes and to develop a tight bound. In this paper, exact expressions for up to seven erasures are developed and a tight bound is proposed to account for eight erasures and above.

The next section introduces the SPC product code model and its parameters. Performance analysis is detailed in section 3. Verification of the analysis and results is done next. The paper concludes by summarizing the main findings of this research.

II. SPC PRODUCT CODE MODEL

Cells that are received in error and cannot be corrected are erased and their location is stored in the recovery system. Without loss of generality, we use cells with the minimum size of one bit. Error detection and erasure location storage is performed through additional coding and headers check.

The cells are arranged into M rows and N column. The recovery process is repeated iteratively rows then columns and so on until no cells can be recovered. The performance of the recovery system is function of the following parameters:

i = number of lost cells in the matrix before recovery.
A_i = number of unrecoverable patterns with i lost cells.
e_i = average number of cells in error after recovery.
P = cell loss probability.
P_i = post decoding (recovery) cell loss probability.

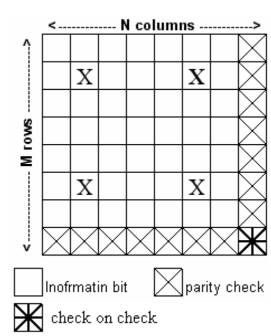

Figure 1. Two-dimensional encoding matrix with unrecoverable 4-error pattern

The post decoding cell loss probability is given by [3]

$$P = \frac{1}{MN}\sum_{i=4}^{MN} e_i A_i p^i (1-p)^{MN-i} \tag{1}$$

The details for finding e_i and A_i for different values of i are discussed in the next section.

III. Performance Analysis

In this part, we derive closed form expressions for e_i and A_i for up to seven errors. An upper bound is developed for more than eight lost cells.

The number of unrecoverable patterns with 1, 2, or 3 errors is zeros because they can be completely recovered. ($A_1 = A_2 = A_3 = 0$). Hence, the summation in (1) starts from 4 rather than from 1.

The only unrecoverable 4-cell pattern occurs when the four cells make a rectangle as shown in the Figure 1. All the four cells are lost. The number of possible unrecoverable patterns is given as

$$A_4 = \binom{M}{2}\binom{N}{2}, \tag{2}$$

$$e_4 = 4.$$

For unrecoverable 5 error patterns, four errors have to make a rectangle shape. The fifth erasure is recoverable and it may occur in any of the remaining (MN-4) choices. Hence,

$$A_5 = A_4\left(MN - 4\right), \tag{3}$$

$$e_5 = 4.$$

A. Recoverability of Six Lost Cells

There can be two scenarios to have six cells unrecoverable patterns. They are either the six are not recoverable or only four are unrecoverable.

Unrecoverable six cells in a rectangle, A6a.

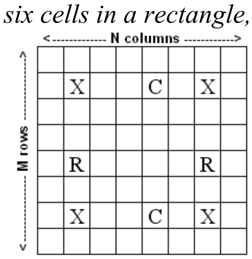

Figure 2. Unrecoverable 6-error rectangular pattern

For all the six cells to be unrecoverable four cells must make a rectangle and the other two are in parallel with rectangle as illustrated in Figure 2. The two in parallel means that they are in front of each other and in the same two rows or two columns where the rectangle is located. To have the other two errors in the same row and in the existing two columns "R case" then we need to choose one row from (M-2) where the two columns are fixed. Alternatively, to have the two errors in the same column and in the existing two rows "C case" the lost cells need to be in one column from (N-2) and we have fixed two rows. However, in this shape, There are three rectangles and any one of them can be considered as the main rectangle. Therefore, we divide by three 3, and the number of patterns is given by

$$A_{6a} = \frac{1}{3}A_4\left[\binom{M-2}{1}+\binom{N-2}{1}\right] = \frac{A_4}{3}\left[M+N-4\right], \tag{4}$$

$$e_{6a} = 6.$$

Unrecoverable Six Cells with a Missing Corner for Two Rectangles, A6b. This scenario occurs when the lost cells form two rectangles sharing a missing erasure. Figure 3 depicts all cases of this scenario for a 3 by 3 matrix. The lost cells indicated by X_1 and X_2 have to be in the same column and any two rows: $\binom{N}{1}\binom{M}{2}$. The cells represented by X_3 and X_4 are in the same row and in a row and columns not used by X_1 and X_2. The total number of scenarios for X_3 & X_4 are $\binom{M-2}{1}\binom{N-1}{2}$. The cells X_5 and X_6 can be located in any of the two remaining diagonals. The total number of possible scenarios is given in (5) where we need to divide by three because the pairs of lost cells can be interchanged.

$$A_{6b} = \frac{1}{3}\binom{M}{2}\binom{N}{1}\binom{M-2}{1}\binom{N-1}{2}2, \tag{5}$$

$$= A_4\frac{2}{3}(M-2)(N-2),$$

$$e_{6b} = 6.$$

where we have used the following identity

$$\binom{k}{1}\binom{k-1}{2} = \binom{k}{2}\binom{k-1}{1},$$

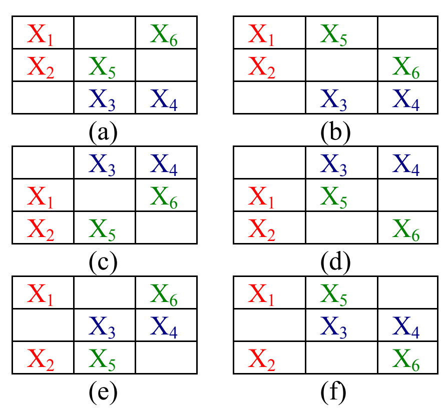

Figure 3. Unrecoverable 6-error with two rectangles missing a shared corner

Two Out of Six Cells are Recoverable, A6c. To be able to recover two cells in errors, then they can be in any where except the four in rectangle and they should not be in the case A_{6a} ("R nor C"). So,

$$A_{6c} = \binom{N}{2}\binom{M}{2}\left[\binom{MN-4}{2} - (M+N-4)\right], \qquad (6)$$

$$e_{6c} = 4.$$

Therefore, the total number of unrecoverable patterns is the sum of A_{6a}, A_{6b}, and A_{6c}. While the number of cells in error after decoding is the average of the three cases and it is given below.

$$A_6 = A_4\left[\binom{MN-4}{2} + \frac{2}{3}(M-2)(N-2) - \frac{2}{3}(M+N-4)\right], \qquad (7)$$

$$e_6 = \frac{1}{A_6}\left[A_{6a}e_{6a} + A_{6b}e_{6b} + A_{6c}e_{6c}\right].$$

B. Recoverability of Seven Cells

There can be three scenarios to have unrecoverable patterns with seven erasures. They are all the seven will remain in error, six will remain in error and one is recovered, and four remain in error and three are recovered.

Seven Unrecoverable Cells, A7a. All the seven cells will be lost if six cells are located as in A_{6b} and the 7th is filling a missing corner and it can be in any of three locations as illustrated by Figure 4.

$$A_{7a} = 3A_{6b}, \qquad (8)$$

$$e_{7a} = 7.$$

An alternative way to get the same result is shown in Figure 5. The main four cells must make a rectangle and the other three cells form a rectangle with one of the main four cells, i.e. the four Xs form the main rectangle and *k*, *m*, *s*, and one X form a second rectangle. For *k*, it needs to be in a free column and in one of the two occupied rows by the main rectangle. For *s*, it is the opposite of *k*. Then for *m*, it has only one place. After that we need to divide by two (2) because in this shape we have two rectangles and any one of them can be the main rectangle which results in the same expression as in (8).

Six in Lost and One Recoverable, A7b. The six cells in errors have the same pattern of A_{6a} while the recoverable cell "k" can be any where else rather the six cells (*MN*-6). Also, the six cells in errors have the same pattern of A_{6b} and the 7th cell can be any where else except these six cells and the three missing corners.

$$A_{7b} = A_{6a}(MN-6) + A_{6b}(MN-9) \qquad (9)$$

$$e_{7b} = 6$$

Four Lost and Three are Recoverable, A7c. To be able to recover three cells in errors, then they can be in any where except the four in rectangle $\binom{MN-4}{3}$ and they should not be in the above two cases.

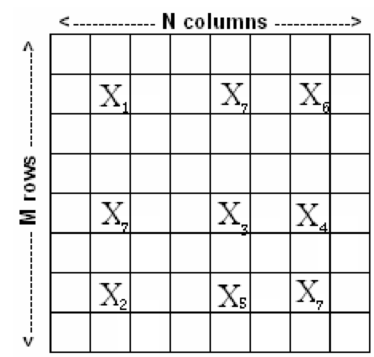

Figure 4. Six cells as in A_{6b} and the 7th

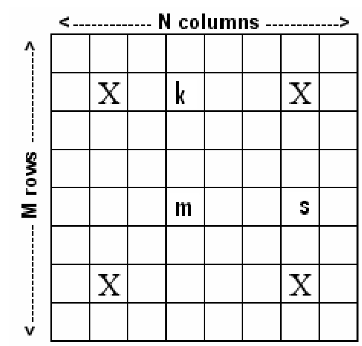

Figure 5. Another way to look at 7 cell

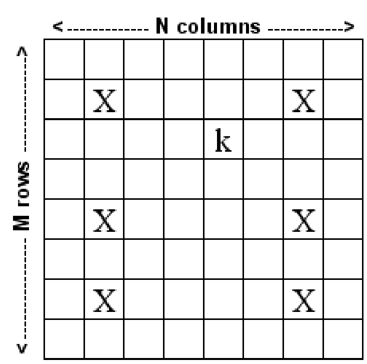

Figure 6. Six in errors and one recoverable

$$A_{7c} = A_4\left[\binom{MN-4}{3} - 4(N-2)(M-2) - (M+N-4)(MN-6)\right] \quad (10)$$

$$e_{7c} = 4$$

Therefore, the total number of unrecoverable patterns is the sum of A_{7a}, A_{7b}, and A_{7c}. While the number of cells in error after decoding is the average of the three cases and it is given below.

$$A_7 = A_{7a} + A_{7b} + A_{7c} \quad (11)$$

$$e_7 = \frac{1}{A_7}(A_{7a}e_{7a} + A_{7b}e_{7b} + A_{7c}e_{7c})$$

C. Recoverability of Eight Cells and Above

For a pattern to be unrecoverable (either all the cells or some), there must be three cells forming a right-angle shape. Recoverability analysis for higher numbers of erasures is tedious and we resort to using upper bound on the performance. We assume that all erasure patterns are unrecoverable and it is further assumed that in this case not all the cells will be recovered. For this upper bound,

$$A_i = \binom{MN}{i} \quad (12)$$

$$e_i = i \quad , \quad 8 \le i \le MN$$

IV. Results & Analysis

To verify the previous expressions an exhaustive simulation was performed where all the possible permutation for a specific number of erasures are generated and decoded. The number of lost cells after recovery agrees with the developed algorithms for all examined scenarios. Table1 summarizes the result for different values of M and N. Those expressions can be used to judge the performance of SPC product codes.

V. Conclusion

Mathematical expressions for the recoverability of up to seven errors were derived and verified. In addition, an upper bound limits the number of unrecoverable patterns for the case of more than seven errors.

Acknowledgment

The authors acknowledge King Fahd University of Petroleum and Minerals, KFUPM, for supporting this research.

TABLE 1: SUMMARY OF POST DECODING LOST CELLS.

M*N	***i***	***e***	***A***
3*3	4	4	9
	5	4	45
	6	4.2857	84
	7	6.5	36
3*4	4	4	18
	5	4	144
	6	4.1707	492
	7	4.7273	792
4*4	4	4	36
	5	4	432
	6	4.1212	2376
	7	4.4314	7344
4*5	4	4	60
	5	4	960
	6	4.0939	7240
	7	4.313	32720
5*5	4	4	100
	5	4	2100
	6	4.0755	21200
	7	4.2417	132300

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