Equalization

Contents

Introduction

In practice for high-speed communication the channel $C(f)$ is not known -with sufficient precision- to design optimum $G_T(f)$& $G_R(f)$. For example, for dial-up telephone line,$C(f)$, changes with the dialed # (route). Wireless and acoustic channels are even worse as they are time-variant.

In such cases we may design $G_R(f)$& $G_T(f)$ such that

$$G_T(f) = \sqrt{X_{rc}(f)}\; e^{-j2\pi f t_0} \qquad |f| \le W$$

$$G_R(f) = \sqrt{X_{rc}(f)}\, e^{-j2\pi f t_1} \qquad |f| \le W$$

Objective: design a receiver that compensate for the channel (distorting +AWGN) which is not known a priori to reduce ISI. This compensation is known as Equalizer**. Equalizers** are filters with adjustable parameters to compensate for the chancel distortion. Reducing the complexity is an objective.

In this module we cover: Maximum Likelihood Sequence Equalizer (MLSE) optimum Pe, Linear equalizers, decision Feedback Equalizers (DFE), and their performance.

Classification of equalizers based on

Adjustment of Coefficients

Preset Equalizers: When the channel impulse response is unknown but time-invariant over the time of transmission (dial-up systems). The channel characteristics may be measured and used to adjust the parameters of the equalizer. (The equalizer coefficients are fixed over the transmission period).

Adaptive Equalizers: The channel is time-variant. The equalizer updates the parameters on periodic basis.

Structure

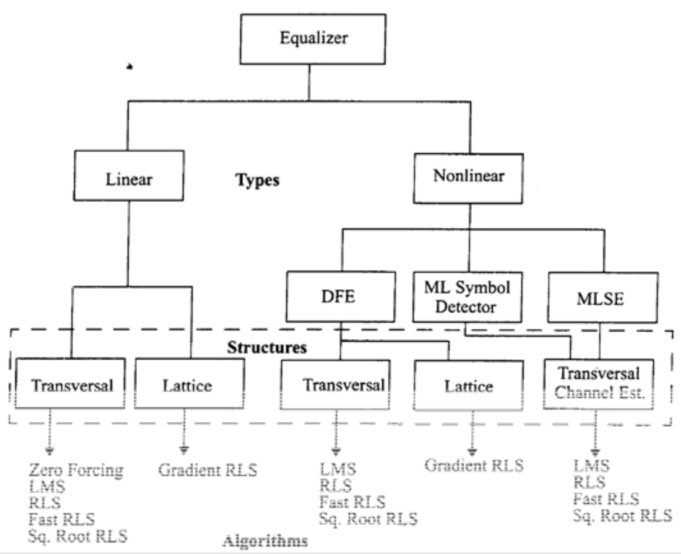
Transversal (tapped-delay-line), Lattice, systolic

Criteria

MLSE, ZF, LMS, RLS (Kalman), DFE.

Training

Trained, Blind, Semi-blind



Optimum Receiver for Channels with ISI and AWGN

Under ISI and AWGN what is the optimum demodulator? Is it still the matched filter?

The answer is yes. We can prove that by showing that the samples at the output of the matched filter are sufficient statistics for estimating $\{I_n\}$.

What is the optimum detector?

It is the maximum likelihood sequence detector as we have seen earlier (for proof see the textbook).

Channel modeling

Let $x(t)$ be the impulse response of the overall channel.

If the channel is distortionless, then the value of $x(mT)$ must be impulse: 0 0 0 0.....0 0 0 1 0 0 0... 0 0 0 0

In the presence of distortion, the channel sampled impulse response is given by (for example)

x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3
0	0.05	0.1	0.2	0.6	1	0	-0.85	0.5

A convenient mathematical representation of the sampled impulse response is the z-transform

$$X(z) = 0.05z^4 + 0.1z^3 + 0.2z^2 + 0.6z^1 + 1 - 0.85z^{-2} + 0.5z^{-3}$$

where z^{-1} represents one unit delay. Note that we can control the delay by multiplying by z^{-delay}.

Example : if the channel impulse response is given by

0	0.33	1	0.5	-0.2	-0.1	0.08	0

What is the output if the transmitted sequence is

1	1	-1	1	-1	-1	1

The output can be found using discrete convolution

0.33	1	0.5	-0.2	-0.1	0.08						
	0.33	1	0.5	-0.2	-0.1	0.08					
		-0.33	-1	-0.5	+0.2	+0.1	-0.08				
			0.33	1	0.5	-0.2	-0.1	0.08			
				-0.33	-1	-0.5	+0.2	+0.1	-0.08		
					-0.33	-1	-0.5	+0.2	+0.1	-0.08	
						0.33	1	0.5	-0.2	-0.1	0.08
0.33	1.33	1.17	-0.37	-0.13	-0.65	-1.19	0.52	0.88	-0.18	-0.18	0.08

It is reasonable to assume that ISI affects a finite number of symbols.

Channel Modeling with Noise Whitening

The overall response $X(z)$ can be represented as the concatenation of two things: the channel $F(z)$, and the noise-whitening filter $\frac{1}{F^*(Z^{-1})}$. In that case the zero forcing equalizer must be $\frac{1}{F(z)}$.

The cascade of the noise-whitening filter and the zero forcing equalizer $C'(z) = \frac{1}{F(z)F^*(z^{-1})} = 1/X(z)$.

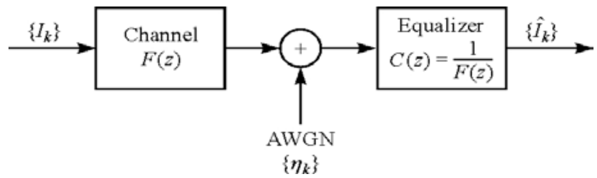


Figure: Block diagram of channel with zero- forcing equalizer.

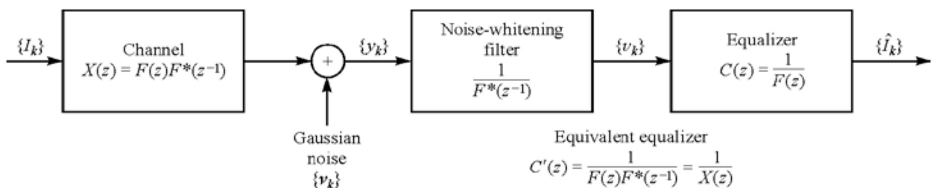


Figure: Block diagram of channel with equivalent zero-forcing equalizer.

The output will be corrupted with white noise.

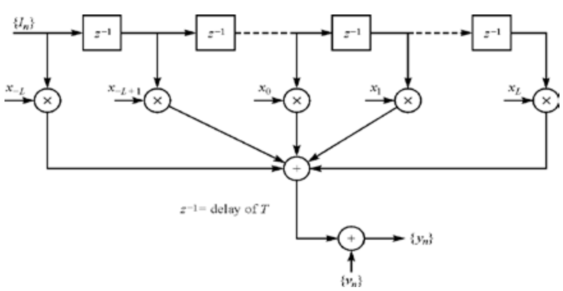


Figure Equivalent discrete-time model of channel with intersymbol interference.

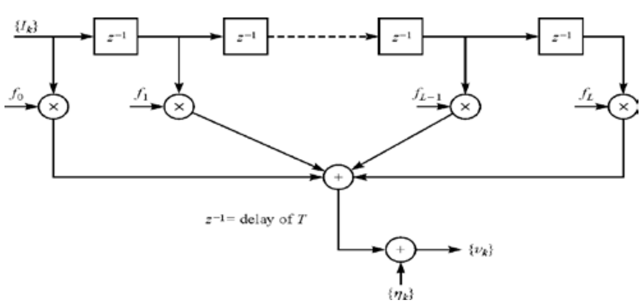
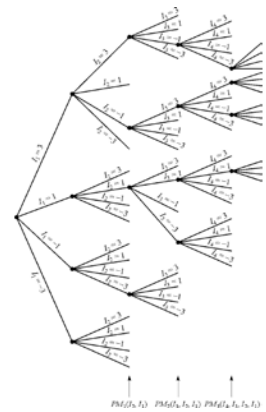


Figure Equivalent discrete-time model of intersymbol interference channel with AWGN.

Maximum Likelihood Sequence Equalizer (MLSE)

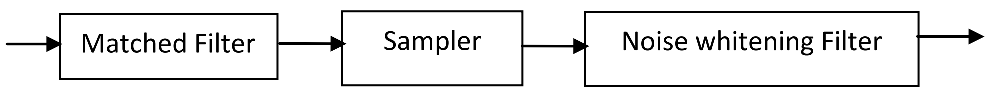


- MLSE complexity grows exponentially with channel time dispersion (memory).
- For each symbol M^{L+1} computed metrics and M^Lsurvivors are kept.
- Though very complicated, the performance of MLSE serve as a benchmark for comparison.

Linear Filters (transversal filter)

Linear complexity with channel dispersion "L"

Recall that the input to the equalizer is the output of the matched filter, we will assume the noise is uncorrelated using WMF (Whitened matched filter)



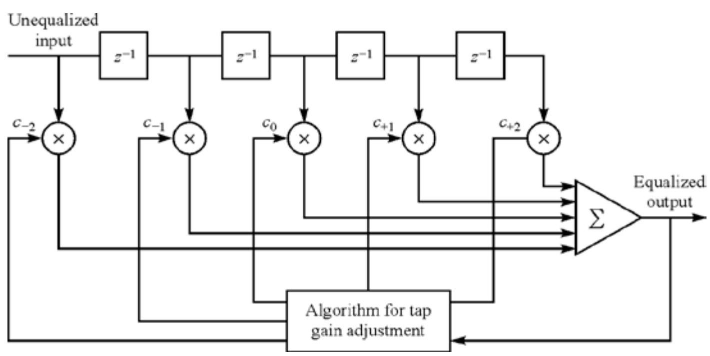


Figure : Linear transversal filter

The output is the estimate to the information sequence $\{I_k\}$

$$I_k = \sum_{j=-k}^{k} c_j v_{k-j}$$

We are ignoring the quantization effect $\widetilde{I_k}$

$\{c_j\}$ are the $2k+1$ tap weight coefficients

How to select the tap coefficients, $\{c_j\}$?

The most meaningful measure of the performance in digital communications is the Average Probability of Error, Pe. However, the relation between P_e& $\{c_j\}$ is computationally complex. Two other criteria are used:

1. Peak distortion criteria
2. MSE criteria

Peak Distortion Criteria

Minimize the worst case intersymbol interference at the output of the equalizer $C'(z) = 1/X(z)$.

The cascade of the discrete-time linear filter having an impulse response of $\{f_n\}$ and the equalizer with an impulse response $\{c_n\}$ can be represented as a single filter with the following output.

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j}$$

The *peak distortion*

$$D = \sum_{n=-\infty, n\neq 0}^{\infty} |q_n| = \sum_{n=-\infty, n\neq 0}^{\infty} \left| \sum_{j=-\infty}^{\infty} c_j f_{n-j} \right|$$

The peak distortion criteria neglects noise.

We will only consider the finite length case.

The peak distortion criteria has been shown to possess a global minimum and no local minima, we can use the method of steepest descent (to be explained later)

The solution to the minimization problem is known if the ***normalized peak distortion criteria*** is less than 1 (open eye diagram) at the input of the equalizer.

$$D_0 = \frac{1}{|f_0|} \sum_{n=1}^{L} |f_n|$$

The solution is the ZFE with $q_0 = 0 \; for \; 1 \leq |n| \leq k \; adn \; q = 0 \; for \; n = 0$

At the output of the equalizer, notice that $\{q_n\} \; for \; k+1 \leq n \leq k+L+1$ are nonzero (Residual)

Minimum Mean Square Error (MSE) Criteria

The MMSE criteria accounts for noise and $\{c_j\}$ coefficients are adjusted to minimize the mean square value of the error,

$$\boldsymbol{J = E|\varepsilon_k|^2 = E\left|I_k - \widehat{I_k}\right|^2}$$

$C'(z) = \frac{1}{X(z)+N_0}$.

Compare the difference between the two criteria for N_0 small and for large values?

Zero Forcing Equalizer (Peak Distortion Criteria)

The idea of zero forcing equalizer (ZFE) is to design the coefficients such that to eliminate the ISI term, that is to force ISI to zero $y_m = x_0 I_m + \sum_{m \neq n} I_n x_m - n + v_m$

From the frequency domain point of view, the overall spectrum of the received signal before detection, which is supposed to be $X_{rc}(f)$ for zero ISI.

$$G_T(f)C(f)G_R(f)G_E(f) = X_{rc}(f)E^{-j2\pi f t_0}$$

We designed $G_T(f)$ & $G_R(f)$ such that

$$G_T(f)G_R(f) = X_{rc}(f)e^{-j2\pi f t_0}$$

The requirement for zero ISI is then $C(f)G_E(f) = 1$ or $G_E(f) = 1/C(f)$ or in the z-domain

$$G_E(z) = \frac{1}{C(z)}$$

Linear ZFE is linear equalizer is a tapped delay line filter. For symmetry, the total number of taps is chosen to be $2k + 1$. With weights denoted by $c_{-k}, \dots., c_0, \dots \dots c_k$

Then the impulse response of the equalizer $g_E(t) = \sum_{n=-k}^{k} c_n \, \delta(t - nT)$

Let $q(t)$denote the equalizer response to the input $x(t)$:

$$q(t) = x(t) * g_E(t) = \sum_{n=-k}^{k} c_n \, x(t - nT)$$

At the sampling instants mT

$$q(mT) = \sum_{n=-k}^{k} c_n x(mT - nT)$$

or

$$q_m = \sum_{n=-k}^{k} c_n x_{m-n} \text{.......(1)}$$

Hence, comes the zero forcing criteria:

$$q_m = \begin{cases} 1 & m = 0 \\ 0 & otherwise \end{cases}$$

Note that the $q_m$has $2k + L$ non-zero terms , in general.

This is obvious since $\sum_{n=-k}^{k} c_n x_{m-n}$ repsresents the discrete convolution of C $(2k + 1)$ terms and x (L) terms . But since we have $2k + 1$ equalizer coeffenceits, we can control only $2k + 1$ sampled values of $q(t)$. That is $q_m = \begin{cases} 1 & m = 0 \\ 0 & m = \pm 1, \pm 2, \dots . . \pm k \end{cases}$..........(2)

It is generally impossible to eliminate the ISI completely at the output of the equalizer.

Equation (1) ,under constrains (2)may be put in matrix form :

$$\boldsymbol{XC} = \boldsymbol{q}$$

where $\boldsymbol{X} = \begin{bmatrix} x_0 & x_{-1} & \cdots & x_{-2k} \\ x_1 & x_0 & \cdots & \cdots \\ \vdots & & & \\ x_{2k} & & & x_0 \end{bmatrix}$ a $(2k+1)(2k+1)$matrix $\boldsymbol{C} = \begin{bmatrix} c_{-k} \\ \vdots \\ c_0 \\ \vdots \\ c\_k \end{bmatrix}$ and $\boldsymbol{q} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

Example

The channel sampled response is given by 0.1 , 1, −0.2 corresponding to (x_{-1}, x_0, x_1)

a) Determine the coefficients of a three tap ZFE.

$$\begin{bmatrix} 1 & 0.1 & 0 \\ -0.2 & 1 & 0.1 \\ 0 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving the above matrix equation

$$c_{-1} = -\frac{5}{52}, c_0 = \frac{50}{52}, c_1 = \frac{10}{52}$$

```
% Dr. Ali Hussein Muqaibel
% EE573
% Equalization
clear all
close all
% Zero Forcing Example
X=[ 1       0.1     0;
    -0.2    1       .1;
    0       -0.2    1];
q=[0 1 0]';
C=inv(X)*q
```

- -0.0962 0.9615 0.1923

b) Find q_m (after equalization).

$$q_m = \sum_{k=-1}^{1} c_n x_{m-n}$$

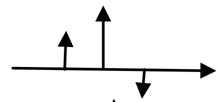
By necessity $q_1 = q_{-1} = 0, q_0 = 1$ (check)

$$q_{-2} = c_{-1}x_{-1} + c_0 x_{-2} + c_1 x_{-3} = -\frac{0.5}{52}$$

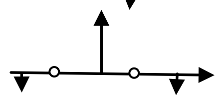
$$q_2 = c_{-1}x_3 + c_0 x_2 + c_1 x_1 = -\frac{2}{52}$$

$L-1$ terms are left nonzero

The impulse response before equalization



After equalization



Tolerance to noise (Error Probability Analysis)

Let us illustrate this by means of the example given above

Remember that for a binary PAM:

$$P_e = Q\left(\sqrt{\frac{d^2}{\sigma^2}}\right) = Q\left(\frac{Signal\ amplitude\ at\ sampling}{\sqrt{noise\ variance}}\right)$$

$$y_m = -0.2I_{m-1} + I_m + 0.1I_{m+1} + v_m$$

What are the possible outcomes, relative to I_m?

I_{m-1}	I_{m+1}	y_m	P
$-I_m$	$-I_m$	$1.1I_m + v_m$	¼
$-I_m$	$+I_m$	$1.3I_m + v_m$	¼
$+I_m$	$-I_m$	$0.7I_m + v_m$	¼
$+I_m$	$+I_m$	$0.9I_m + v_m$	¼

$$P_e = \frac{1}{4}\left\{Q\left(\frac{1.1d}{\sigma_v}\right) + Q\left(\frac{0.7d}{\sigma_v}\right) + Q\left(\frac{1.3d}{\sigma_v}\right) + Q\left(\frac{0.9d}{\sigma_v}\right)\right\}$$

Considering the worst case bound

$$P_e > Q\left(\frac{0.7d}{\sigma_v}\right)$$

After equalization $y_m = I_m + v_m$ (neglecting the residual term) but the noise after equalization is correlated ad and the above is not true.

What is the variance of the noise samples after equalization? σ_u^2

$$u_m = \sum_{n=-k}^{k} c_n v_{m-n}$$

$$E[u_m] = \sum_{n=-k}^{k} c_n E[v_{m-n}] = 0$$

$$\sigma_u^2 = E[u_m^2] = \sum_{n=-k}^{k} \sum_{l=-k}^{k} c_n c_l E[v_{m-n} v_{m-l}]$$

But $E[v_{m-n}] = 0 \; for \; m \neq n$

$$\boxed{\sigma_u^2 = \sum_{n=k}^{k} c_n^2 E[v_{m-n}^2] = \sigma_v^2 \sum_{n=-k}^{k} c_n^2}$$

For the above example $\sigma_u^2 = 0.9\sigma_v^2$

$$P_e = Q\left(\frac{d}{\sqrt{0.9}\sigma_v}\right) = Q\left(\frac{1.05d}{\sigma_v}\right)$$

This might look better than the ideal channel but remember that the channel taps are not normalized (sum of square of channel taps should be 1) and alos the residual is ignored.

Exercise: check the peak distortion criteria before and after equalization for the previous example

Recall from the introduction to ZFE, even though the equalizer eliminates the ISI, but the noise variance at the output of the ZFE is , in general, higher than the noise variance at the output of the optimum receiver filter $|G_R(f)|$ for the case in which the channel is known.

Peak distortion criteria:

Weakness of ZFE

By forcing the ISI to zero (nearly), the ZFE is in essence minimizing the peak distortion:

$$D_0 = \frac{1}{|q_0|} \sum_{m \neq 0} |q_m|$$

Two points must be emphasized about the zero forcing algorithm:

The algorithm is only optimum in the sense of minimizing D, only if $D_0 < 1$.

The peak distortion criteria and hence the ZFE neglects the noise.

For the previous example

$$D_0 \; before \; equalization = 0.3$$

$$D_0 \; after \; equalization = 0.05$$

The weakness of ZFE is it focuses on ISI and ignores the presence of noise. As a results its use ,ay results in significant noise enhancement. We have stated before that $G_E(f) = \frac{1}{C(f)}$. In the range of frequencies where $C(f)$ is small, the equalizer compensates by lacing a large gain in that frequency range, thus greatly enhance the noise. Or from another view point $\sigma_n^2 = \sigma^2 \sum_{n=-k}^{k} c_n^2$

If $c_n > 1$ the noise variance is increased and degrades the performance of the system. The ZFE is suitable for channels with non-sever distortion.

What should be the best criteria for optimizing the filter coefficients? Since the performance of the digitally modulated communications is measured by the average probability of error is highly a nonlinear function of $\{c_n\}$. A criterion that looks (and proves) to be a closer to optimum is the one that considers minimizing the combined ISI-and-noise effects at the output of the equalizer. It is called the means square error (MSE).

In class practice:

Binary PAM is used to transmit information over un-equalized linear filter channel. When a=1 is transmitted, the noise-free output of the modulator is x_m

$$x_m = \begin{cases} 0.15 & m = 1 \\ 0.9 & m = 0 \\ 0.15 & m = -1 \\ 0 & otherwise \end{cases} \qquad c_m = \begin{cases} -0.1 & m = 1 \\ 1.2 & m = 0 \\ -0.1 & m = -1 \end{cases}.$$

a linear equalizer was designed by Mr. XYZ with coefficients c_m. Mr. XYZ is consulting you to assess his equalizer.

i. Evaluate the equalized system. What is your final recommendation?

ii. For the un-equalized system, what is the sequence/s that will lead to the worst case interference, and what is its probability?

Comment about the ZFE

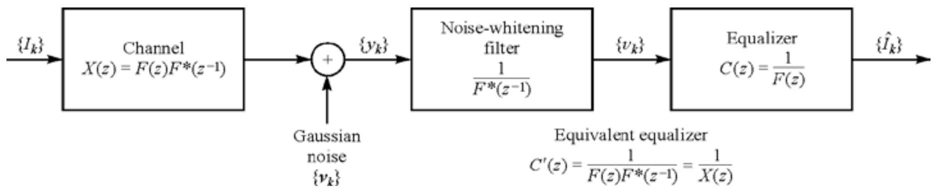


Figure: Block diagram of channel with equivalent zero-forcing equalizer.

It should be remembered, however, that the P_e for the equalizer output is in general, higher than that of the optimum receiver when the channel is known and time invariant. Where

$$\sigma_n^2 = \int_{-\infty}^{\infty} S_n(f)\,|G_R(f)|^2 df = \int_{-\infty}^{\infty} S_n(f)\frac{|X_{rc(f)}|}{|C(f)|}\ df$$

Because $|G_R(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}$. For the equalized case

$$\sigma_n^2 = \int_{-\infty}^{\infty} S_n(f)\,|G_R(f)|^2 |G_R(f)|^2 df = \int_{-\infty}^{\infty} S_n(f)\frac{|X_{rc(f)}|}{|C(f)|^2}\ df$$

Because $|G_R(f)| = \sqrt{|X_{rc}(f)|}$ and $G_E(f) = \frac{1}{C(f)}$ for zero forcing equalizer.

Usually $|C(f)| < 1$ for some range of frequencies for ZFE.

The noise variance at the output of the ZFE is in general higher compared with the optimum receiver when the channel is known.

Mean Square Error Equalizer

$\{\boldsymbol{c_j}\}$ are adjusted to minimize $\boldsymbol{\varepsilon_k = I_k - \widehat{I_k}}$. The performance index

$$\boldsymbol{J = E|\varepsilon_k|^2 = E\left|I_k - \widehat{I_k}\right|^2}$$

$$I_k \ is \ the \ k^{th} information \ symbol$$

$$\widehat{I_k} \ is \ the \ k^{th} estimate \ after \ the \ equalizer$$

Finite-length Equalizer

The output of the equalizer

$$\widehat{I_k} = \sum_{j=-k}^{k} c_j v_{k-j}$$

The MSE for the equalizer with $2k + 1$ taps

$$J(k) = E\left|\boldsymbol{I_k} - \widehat{I_k}\right|^2 = E\left|\boldsymbol{I_k} - \sum_{j=-k}^{k} c_j v_{k-j}\right|^2$$

Because $\varepsilon_k = I_k - \widehat{I_k}$

$$J(k) = E|\varepsilon_k|^2 = E(\varepsilon_k I_k^*) - E\left(\varepsilon_k \widehat{I}_k^*\right)$$

To minimize $J(k)$ we invoke the orthogonality principle in mean square estimation.

i.e we select the coefficient $\{c_j\}$ to render the error orthogonal to the signal sequence $\{v_{k-1}^*\}$

$E(\varepsilon_k v_{k-1}^*) = 0 \qquad -\infty < l < \infty$ in case of infinite length

Going through detailed proof (see textbook) we conclude that:

$$\sum_{j=-k}^{k} c_j \Gamma_{l_j} = \xi_l \ \ where - k < l < k$$

$$\Gamma_{l_j} = \begin{cases} x_{l-j} + N_0 \delta_{lj} & |l-j| \le L \\ 0 & otherwise \end{cases}$$

$$\xi_l = \begin{cases} f_{-l}^* & -L \le l \le 0 \\ 0 & otherwise \end{cases}$$

In matrix Format $\boldsymbol{\Gamma C = \xi}$

C: column vector $2k + 1$ taps

Γ: $(2k + 1)(2k + 1)$ Hermitian covariance matrix Γ_{ij}

In mathematics, an **Hermitian matrix** (or **self-adjoint matrix**) is a square matrix with complex entries that is equal to its own conjugate transpose – that is, the element in the *i*-th row and *j*-th column is equal to the complex conjugate of the element in the *j*-th row and *i*-th column, for all indices *i* and *j*: $a_{ij} = \overline{a_{ij}}$.

If the conjugate transpose of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A^T}$, then the Hermitian property can be written concisely as $\boldsymbol{A} = \boldsymbol{A^T}$

For example, $\begin{bmatrix} 5 & 3+j \\ 3-j & 1 \end{bmatrix}$…………………..Reference Wikipedia

ξ: column vector $2k+1$ $\quad \xi_i$

$$C_{opt} = \Gamma^{-1}\xi$$

The associated minimum value of $J(k)$

$$J_{min}(k) = 1 - \sum_{j=-k}^{0} c_j f_{-j} = 1 - \xi^{+*}\Gamma^{-1}\xi$$

And $\quad \gamma = \frac{1-J_{\min}}{J_{\min}}$ SNR

$$\begin{pmatrix} x_0+N_0 & x_{-1} & \\ x_1 & x_0+N_0 & x_{-1} \\ & & x_0+N_0 \end{pmatrix}\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} f_1^* \\ f_0^* \\ f_{-1}^* \end{pmatrix}$$

Example

Binary PAM is used to transmit info. Over an un-equalized linear filter channel. When $a = 1$ is transmitted, the noise-free output of the demodulator is $x_m = \begin{cases} 0.3 & m = 1 \\ 0.9 & m = 0 \\ 0.3 & m = -1 \\ 0 & otherwise \end{cases}$

Design a 3-tap MSE equalizer.

Assume that the noise power spectral density is 0.1 W/Hz.

A discrete time transversal filter equivalent to the cascade of the trasmitting filter $g_T(t)$, the channel $c(t)$, the matched filter at the receicer $g_R(t)$ and the sampler, has tap gain coefficients $\{x_m\}$, where :

$$x_m = \begin{cases} 0.9 & m=0 \\ 0.3 & m=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The noise ν_k, at the output of the sampler, is a zero-mean Gaussian sequence with autocorrelation function :

$$E[\nu_k \nu_l] = \sigma^2 x_{k-l}, \qquad |k-l| \leq 1$$

If the $\mathcal{Z}$-transform of the sequence $\{x_m\}$, $X(z)$, assumes the factorization :

$$X(z) = F(z)F^*(z^{-1})$$

then the filter $1/F^*(z^{-1})$ can follow the sampler to white the noise sequence ν_k. In this case the output of the whitening filter, and input to the MSE equalizer, is the sequence :

$$u_n = \sum_k I_k f_{n-k} + n_k$$

where n_k is zero mean Gaussian with variance σ^2. The optimum coefficients of the MSE equalizer, c_k, satisfy :

$$\sum_{n=-1}^{1} c_n \Gamma_{kn} = \xi_k, \qquad k = 0, \pm 1$$

where :

$$\Gamma(n-k) = \begin{cases} x_{n-k} + \sigma^2 \delta_{n,k}, & |n-k| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_(k) = \begin{cases} f_{-k}, & -1 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

With

$$X(z) = 0.3z + 0.9 + 0.3z^{-1} = (f_0 + f_1 z^{-1})(f_0 + f_1 z)$$

we obtain the parameters f_0 and f_1 as :

$$f_0 = \begin{cases} \pm\sqrt{0.7854} \\ \pm\sqrt{0.1146} \end{cases}, \qquad f_1 = \begin{cases} \pm\sqrt{0.1146} \\ \pm\sqrt{0.7854} \end{cases}$$

The parameters f_0 and f_1 should have the same sign since $f_0 f_1 = 0.3$. However, the sign itself does not play any role if the data are differentially encoded. To have a stable inverse system $1/F^*(z^{-1})$, we select f_0 and f_1 in such a way that the zero of the system $F^*(z^{-1}) = f_0 + f_1 z$ is inside the unit circle. Thus, we choose $f_0 = \sqrt{0.1146}$ and $f_1 = \sqrt{0.7854}$ and therefore, the desired system for the equalizer's coefficients is

$$\begin{pmatrix} 0.9+0.1 & 0.3 & 0.0 \\ 0.3 & 0.9+0.1 & 0.3 \\ 0.0 & 0.3 & 0.9+0.1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.7854} \\ \sqrt{0.1146} \\ 0 \end{pmatrix}$$

Solving this system, we obtain

$$c_{-1} = 0.8596, \quad c_0 = 0.0886, \quad c_1 = -0.0266$$

Performance Characteristics of the MSE Equalizer

Both minimum MSE & Probability of error are two performance measures.

Probability of Error Performance of the Linear MSE Equalizer

There are some equations for the infinite length equalizer for MSE J_{min} and output SNR γ

Without proof

$$J_{min} = \frac{T}{2\pi} \int_{-\pi/T}^{\frac{\pi}{T}} \frac{N_0}{T^{-1} \sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 + N_0}$$

If no ISI $J_{min} = \frac{N_0}{N_0+1}$ $\qquad 0 \le J_{min} \le 1$

The required SNR for the infinite number of taps case

$$\gamma_\infty = \frac{1 - J_{min}}{J_{min}}$$

Infinite length not practical but helpful (limit)

No simple relation between $J_{min}\ and\ \gamma_\infty and\ P_e$

For the MSE there will be some residual ISI not like the infinity length ZFE.

The residual cannot be represented as an additional Gaussian noise term and cannot be easily represented as P_e.

We can use similar procedure like we did with the ZFE example.

$$\sigma_n^2 = N_0 \sum_{j=-k}^{k} c_j^2$$

To illustrate the performance of MSE see the following three Figures.

Binary (antipodal) using Monte Carlo Simulation

- No ISI is also illustrated.
- Channel *a* in the Figure : Typical response of a good quality telephone line. Channels *b* & *c* represent channels with sever ISI.
- For linear equalizer; the spectrum response of channel *c* is very bad (worst spectral characteristics).
- The error rate for channel *a* is within 3dB of the no interference case.

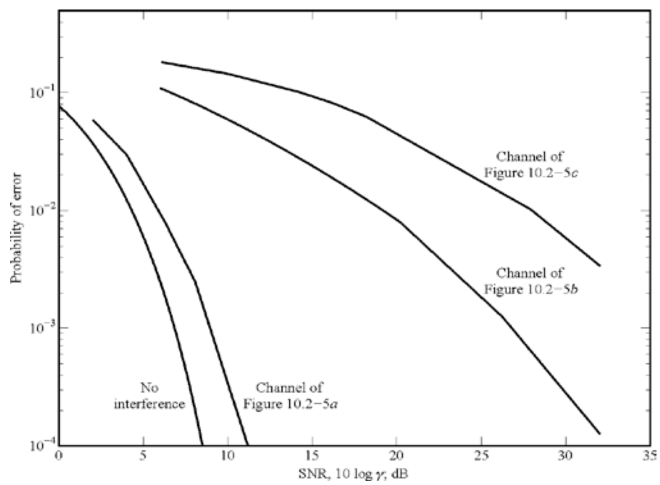


Figure: Error rate performance of linear MSE equalizer. Thirty-one taps in transversal equalizer.

$$\left(\gamma = \frac{1}{N_0}\sum_k |f_k|^2\right)$$

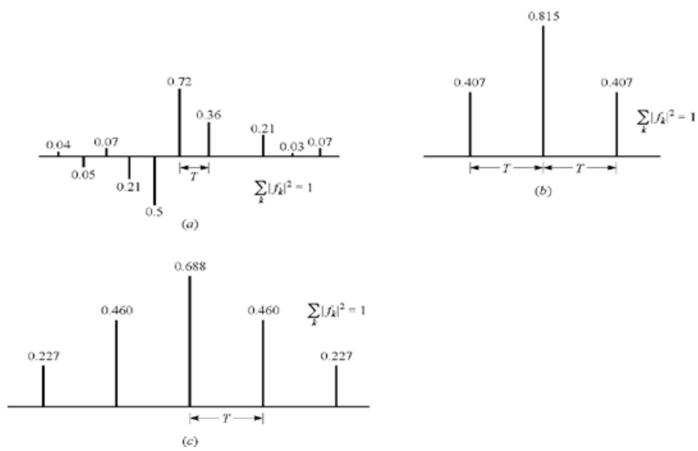


Figure Three discrete-time Channel characteristics.

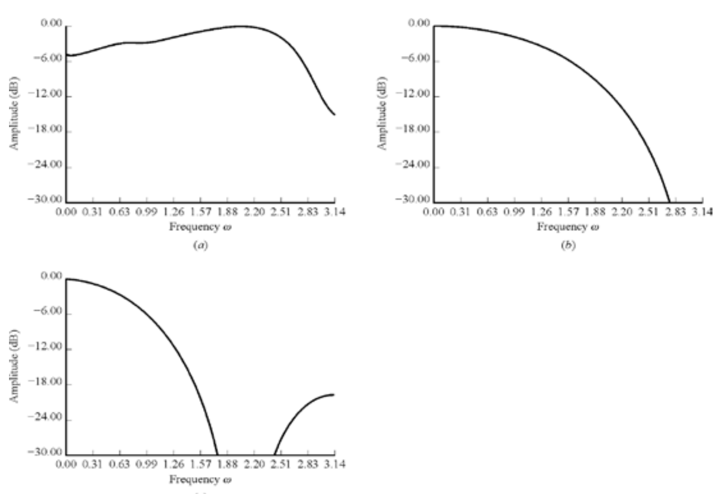


Figure Amplitude spectra for the channels shown in Figure 10.2-5a, b, and c, respectively.

Conclusion about linear equalizers (ZFE,MSE) performance

- Linear equalizers are good for channels with no spectral nulls (telephone lines) (Well behaved channels)
- Linear equalizers are inadequate as compensators for channels with nulls (radio transmission). Nulls results in large noise enhancements.
- This motivates the solution for ISI through nonlinear Decision Feedback Equalizers (DFE).

Decision Feedback Equalizers (DFE)

Motivation

- Linear equalizers (e.g. ZFE & MMSE):
  - Very effective on channels where the ISI is not severe.
  - Consider the three channels examples. MSE is effective for channel *A* but not *C* & *B*.

DFE is nonlinear equalizer that employs previous decision to eliminate the ISI caused by previously detected symbols on the current symbol to be detected.

Operation

- What is the thing that makes DFE nonlinear?
  - The detector.
- The basic idea is that, assuming past decisions are correct, the ISI contributed by these symbols can be completely cancelled by subtracting appropriately weighted past symbols values from the equalizer output.
- Denote the sampled impulse response of the channel by $\{v_k\}$, which extends from $-k_1, \ldots\ldots, k_2$
- The response of the channel to an input sequence $\{I_n\}$ is the convolution:

$$y_m = \sum_{n=-k_1}^{k_2} v_n I_{m-n} = v_0 I_m + \sum_{n=-k_1}^{-1} v_n I_{m-n} + \sum_{n=1}^{k_2} v_n I_{m-n}$$

$$y_m = \textit{desired term} + \textit{ISI due to the previous symbols} + \textit{ISI due to coming symbols}$$

$$I_m = \sum_{n=-k_1}^{k_2} v_n I_{m-n} = \frac{y_m}{v_0} - \sum_{n=-k_1}^{-1} \frac{v_n}{v_0} I_{m-n} - \sum_{n=1}^{k_2} \frac{v_n}{v_0} I_{m-n}$$

$\{v_n\}$ for $n = -k_1, \ldots \ldots \ldots, k_2$ are known (or measured)

$\{I_{m-n}\}$ in the middle term are previous symbols. We can assume that these are the outputs of the decision device assuming the decision is correct.

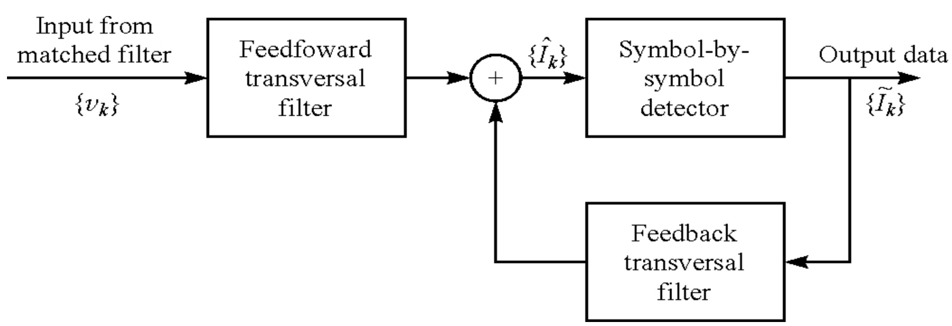


Figure. Structure of decision-feedback equalizer

- $\{\tilde{I}_k\}$ refers to the previously detected symbols. So we may structure the equalizer
- All the previous ISI is cancelled.

- The post cursors will be taken care off by a feed forward filter.
- In general a DFE consists of a feed forward filter section and a feedback filter section, for the following reason.
- The input to the feedback filter is subtracted from the output of the feed forward filter to form the input to the detector.
- Of course, an error in making decision will propagate!

The following figure shows the performance of DFE when used for channel **B**, the performance is greatly improved compared with linear equalizers.

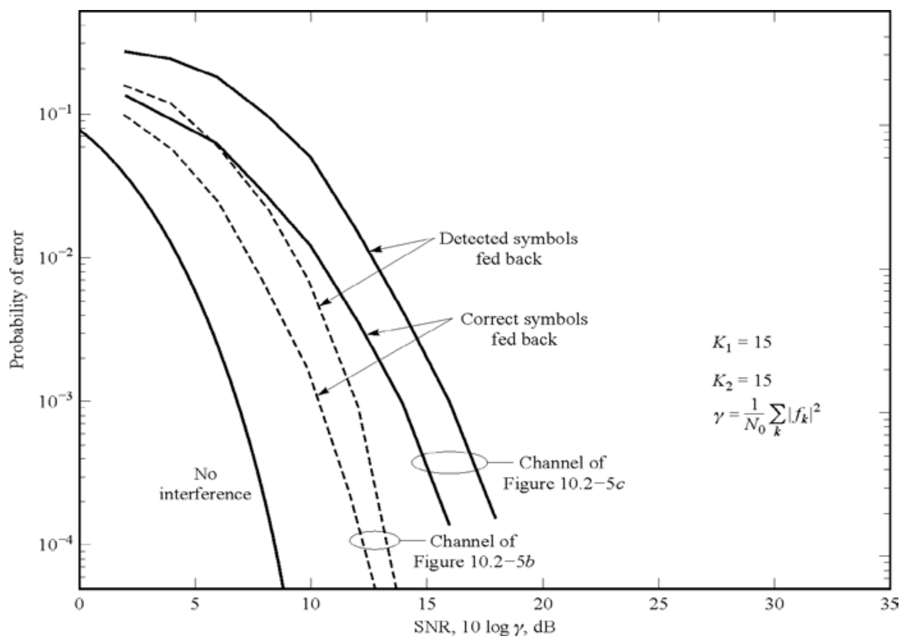


Figure Performance of decision-feedback equalizer with and without error propagation.

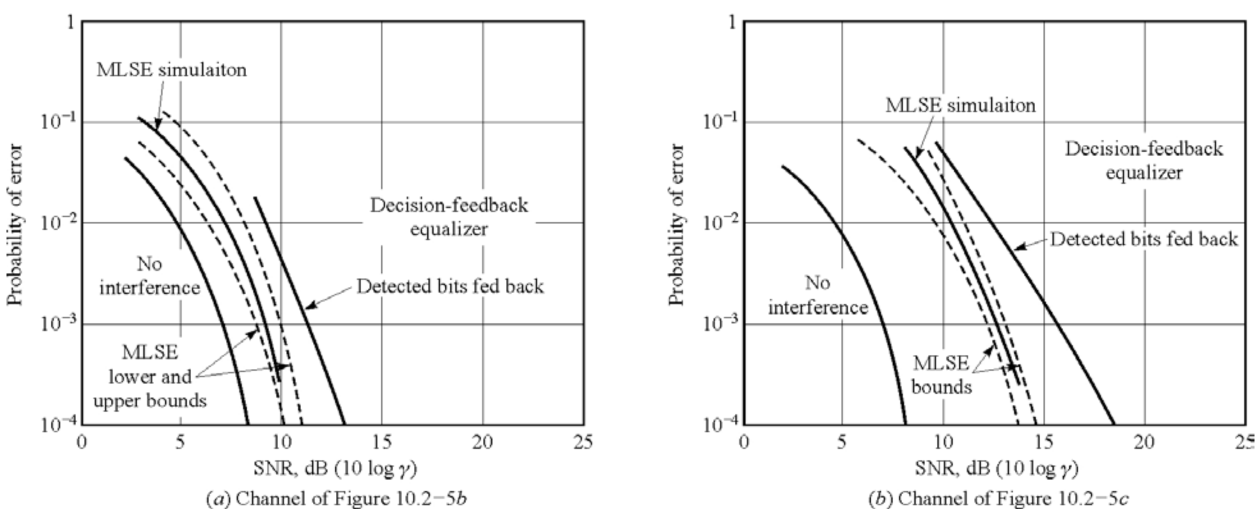


Figure: Comparison of performance between MLSE and decision-feedback equalization for channel characteristics shown (a) channel B (b) in Channel C

Fractionally Spaced Equalizers (FSE) VS Symbol Rate Equalizer (SRE)

Symbol rate equalizer (SRE): the equalizer taps are spaced at 1/T (symbol rate)

Optimum **if** equalizer is preceded a filter matched to the channel distorted transmitted pulse. **If not known** receiver is matched to the transmitted pulse & sampling time is optimized. The overall performance is sensitive to the choice of sampling time.

Fractionally spaced equalizer (FSE): is based on sampling the incoming symbol at least as fast as Nyquist rate.

$F_{max} = \frac{1+\beta}{2T}$. Nyquist rate is twice the highest frequency $2F_{max} = \frac{1+\beta}{T}$, Time is inversely related to frequency and tap spacing is $\frac{T}{1+\beta}$

$$\beta = 1 \; means \frac{T}{2} \; spaced \; equalizer$$

$$\beta = 0.5 \; means \frac{2}{3}T \; spaced \; \text{equalizer}$$

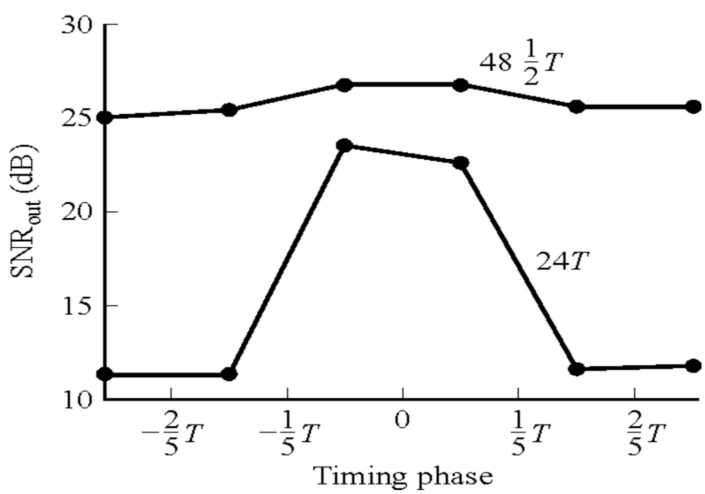
Tap spacing is $\frac{MT}{N}$ where M and N are integers with $N > M$.

The objective of fractionally spaced equalizers is to avoid aliasing.

Optimum FSE is equivalent to the optimum linear receiver consisting of the matched filter (to the distorted signal) followed by a symbol rate equalizer.

Comparison SRE vs FSE

FSE show better performance & less sensitivity to timing phase.



Is the comparison fair? # of taps or spanned time?

Baseband & Passband Equalizers

Practical implementation could be baseband or passband. The advantage of implementing an equalizer in passband is timing and phase advantage)

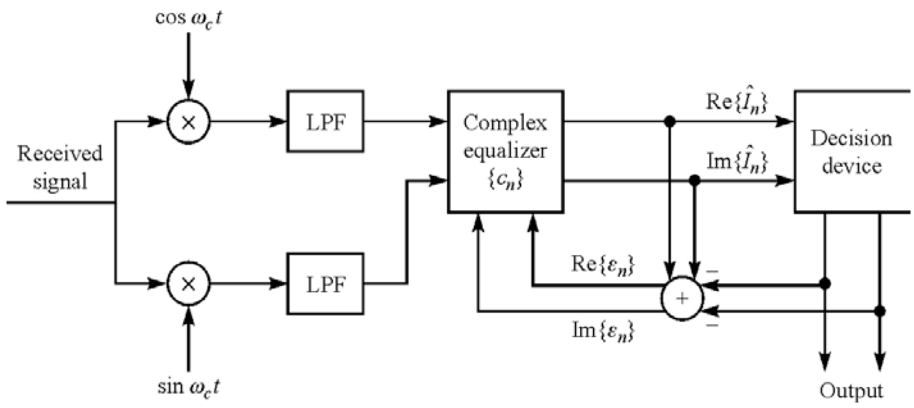


Figure QAM and PSK signal demodulator with baseband equalizer**.**

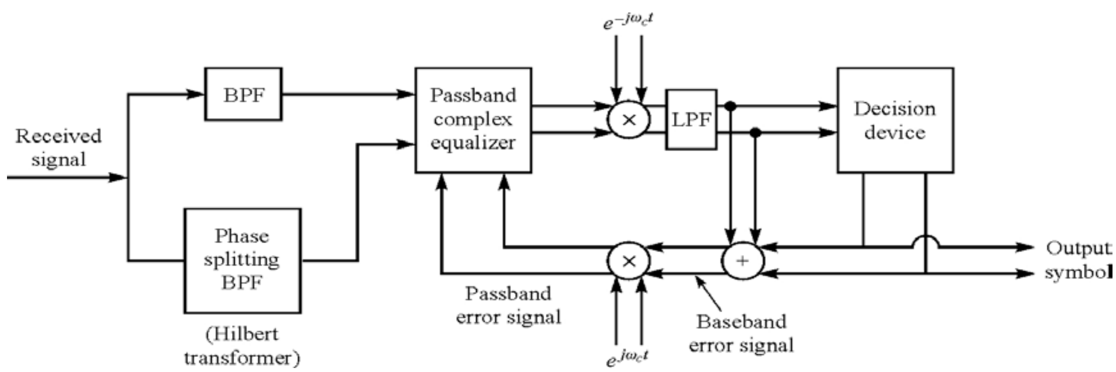


Figure . QAM or PSK signal equalization at passband.

Adaptive Equalization

Motivated by time varying channels.

The tap coefficients of the zero forcing equalizers can be obtained by solving the matrix :

$$\boldsymbol{XC} = \boldsymbol{q}$$

Similar equation can be used to obtain those of MMSE

$$\boldsymbol{\Gamma C} = \boldsymbol{\xi}$$

Let us define the general form:

$$\boldsymbol{BC} = \boldsymbol{d}$$

then

$$\boldsymbol{C_{opt}} = \boldsymbol{B^{-1}d}$$

In practical implementation of equalizers, the optimum coefficients are obtained by an an **iterative** procedure to avoid computing B^{-1}.

This is in particular necessary when the channel is **time-variant**; hence the entries of B and d (for MSE) vary.

The simplest iterative procedure is the method of **Steepest Descent**. The discussion below refers to the **MSE adaptive** equalizer since it is the most **widely used**.

The MSE expression is a **second order** function of the coefficients $C_{-k}, \ldots .., C_0, \ldots \ldots., C_k$. It can be visualized as $(2k+1)$-dimensional bowl shaped surface.

The adaptive process through successive adjustments of tap coefficients has the task of continuously seeking the bottom of the bowl. “shape is changing”

Method of Steepest Descent

(1) Begin by selecting arbitrary the vector C, say C_0.

(2) Find the gradient vector G_0 which is the derivative of the MSE with respect to the $2k+1$ coeffeceints

$$G_0 = BC_0 - d$$

$$G_k = \frac{1}{2}\frac{dJ}{dC_k} = \Gamma C_k - \xi = -E(\varepsilon_k v_k^*)$$

Where $\varepsilon_k = I_k - \widehat{I_k}$

v_k: is the vector of received signal samples that make up the estimates $\widehat{I_k}$

(3) Modify the tap coefficient such that

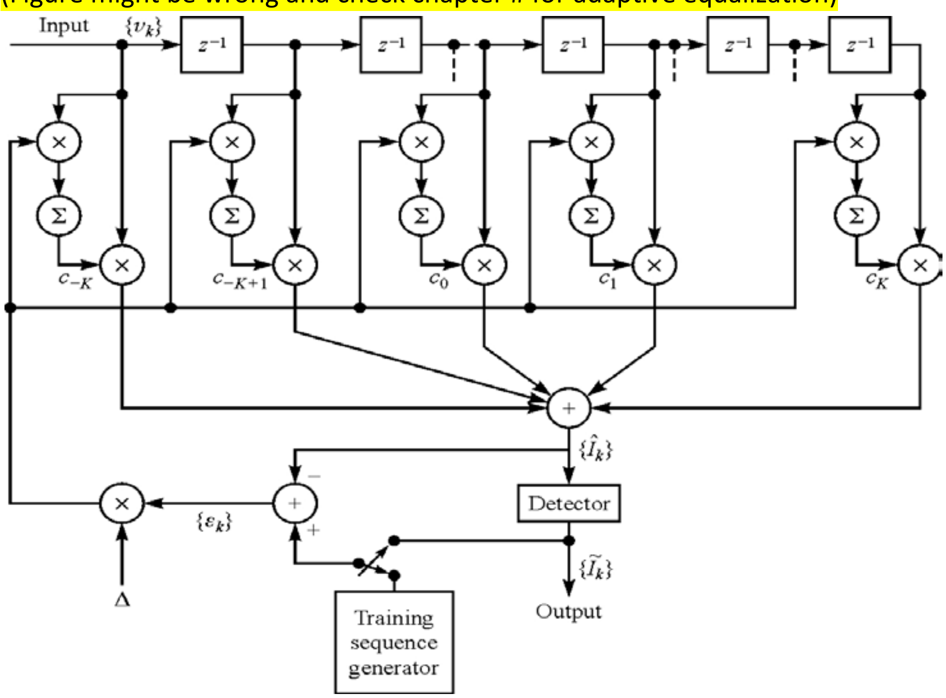
$$C_1 = C_0 - \Delta G_0 \text{ or in general } \boldsymbol{C_{k+1}} = \boldsymbol{C_k} - \Delta \boldsymbol{G_k}$$

Where Δ is the step size parameter for the iterative process. We will discuss its rule later.

(4) Repeat (2) and (3). At the k^{th} iteration, the process continues until $\boldsymbol{G_n} \to \mathbf{0}$ and $\boldsymbol{C_k} \to \boldsymbol{C_{opt}}$, K is the number of iterations.

The equalizer coefficients are thus updated at the symbol rate T

(Figure might be wrong and check chapter # for adaptive equalization)



- Steepest decent algorithm is a gradient based method which employs recursive solution over problem (cost function)

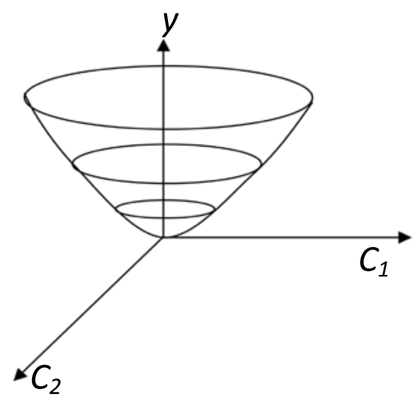
- The current equalizer taps vector is $C(n)$ and the next sample equalizer taps vector weight is $C(n+1)$, We could estimate the $C(n+1)$ vector by this approximation:

$$C[n] = C[n+1] + 0.5\mu(-\nabla J[n])$$

- The gradient is a vector pointing in the direction of the change in filter coefficients that will cause the greatest increase in the error signal. Because the goal is to minimize the error, however, the filter coefficients updated in the direction opposite the gradient; that is why the gradient term is negated.
- The constant μ is a step-size. After repeatedly adjusting each coefficient in the direction opposite to the gradient of the error, the adaptive filter should converge.

Example

Given the following function we need to obtain the vector that would give us the absolute minimum. $Y(c_1, c_2) = C_1^2 + C_2^2$. It is obvious that $C_1 = C_2 = 0$, give us the minimum. **Find the solution by the steepest descend method.**



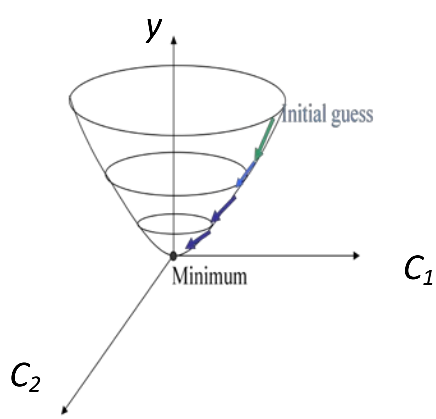
- We start by assuming $(C_1 = 5, C_2 = 7)$
- We select the constant . If it is too big, we miss the minimum. If it is too small, it would take us a lot of time to het the minimum. I would select $\mu = 0.1$.
- The gradient vector is: $\nabla y = \begin{bmatrix} \frac{dy}{dc_1} \\ \frac{dy}{dc_2} \end{bmatrix} = \begin{bmatrix} 2C_1 \\ 2C_2 \end{bmatrix}$
- So our iterative equation is:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n+1]} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n]} - 0.05 * \nabla y = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n]} - 0.1 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n]} = 0.9 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n]}$$

$$Iteration1: \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$Iteration2: \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 6.3 \end{bmatrix}$$

$$Iteration3: \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0.405 \\ 0.567 \end{bmatrix}$$



……

$$Iteration\ 60: \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.013 \end{bmatrix}$$

$$\lim_{n\to\infty} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{[n]} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we can see, the vector [c1,c2] converges to the value which would yield the function minimum and the speed of this convergence depends on .

Training Sequence

The coefficients updating algorithm is based on calculating

$$e_k = I_k - \widehat{z_k}$$

I_k is assumed to be correctly decoded. But in fact we are using $e_k = \widehat{I_k} - z_k$

In practice to help the equalizer to adjust coefficients at the beginning of the process (and from time to time), it is trained by the transmission of a known PN sequence $\{I_k\}$, over the channel. The equalizer is said to be in training mode. After that it shifts to the decision-direct mode.

Step Size

A crucial parameter for the convergence of the algorithm is the step-size.

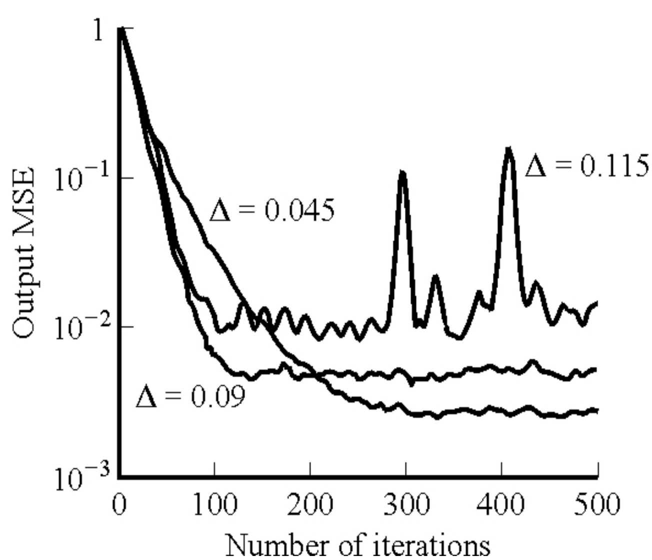


Figure Initial convergence characteristics of the LMS algorithm with different step sizes. [*From* Digital Signal Processing, *by J. G. Proakis and D. G. Manolakis, 1995, Prentice Hall Company. Reprinted with permission of the publisher.*

From the figure we note the following:

$\Delta = 0.115$ cause the algorithm not to converge

$\Delta = 0.09$ It converges in 100 iterations

$\Delta = 0.045$ The convergence is slowed (300 iterations) but a lower MSE is achieved, indicating that the estimated coefficients are close to optimum.

In some research papers, rules of thumbs and limit for selecting the step size are give.

LMS

$$0 < \Delta < 2/\lambda_{max}$$

$|1 - \Delta\lambda_{max}| < 1$ where $\lambda = -k, \ldots\ldots k$

Where $\{\lambda_k\}$ is the set of $2k + 1$ (possibly non-distinct) Eigen values of Γ. $\lambda_k > 0$ for all k

Blind equalization: “no training sequence”

What is semi-blind equalization? (find out)

MSE: mean square error “cost”

MMSE: minimum mean square error “criteria”

LS: least squares “algorithm” requires matrix inversion (non iterative)

MLS: minimum least square “algorithm” iterative, avoid matrix inversion.