Signal Design for Bandlimited Channels

Contents

Introduction and System Model

The transmitted sequence is $v(t) = \sum_0^\infty I_n g(t - nT)$

This is the equivalent low pass of $s(t)$

The received sequence is $r(t) = \sum_0^\infty I_n h(t - nT) + z(t)$

Where $h(t) = c(t) * g(t)$

$$C(f) and\ G(f) = 0\ for\ |f| > W$$

<u>First</u> : design $g(t)$with no channel distortion (only band limitation)

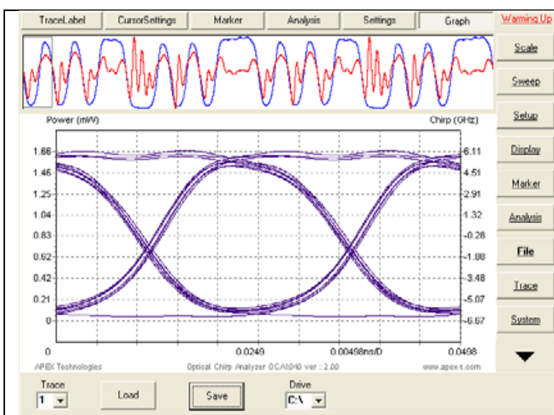
<u>Second</u> : design $g(t)$ with channel distortion.

ISI: is intersymbol interference (time dispersion in the sampling instances) due to band limitation.

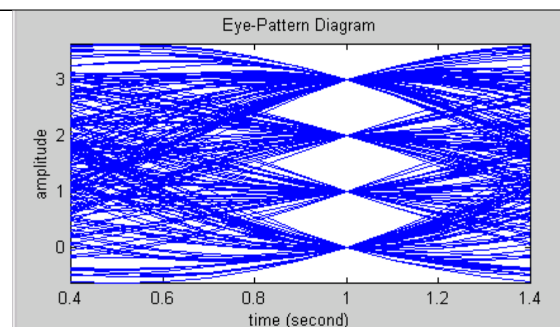
We would like to get zero ISI. We can get that with

1) Signal design if the channel is not distorting.
2) Equalization if the channel is distorting.
3) Adaptive equalization if the channel is time-varying (distorting).

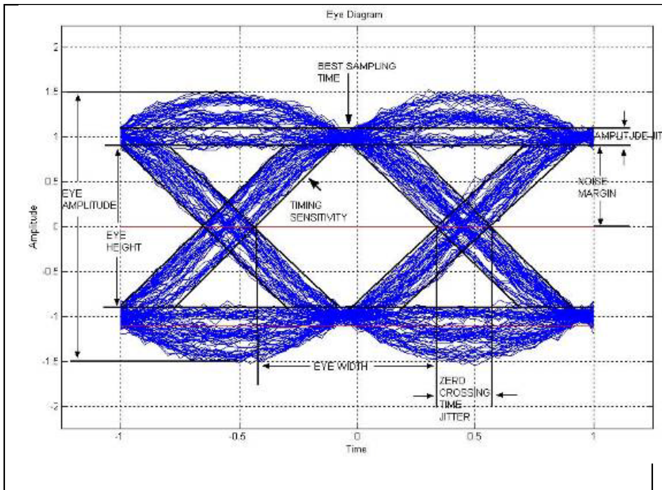
Eye Pattern: Synchronized superposition of all possible realizations of the signal of interest (receiver output) viewed within a particular signaling interval.

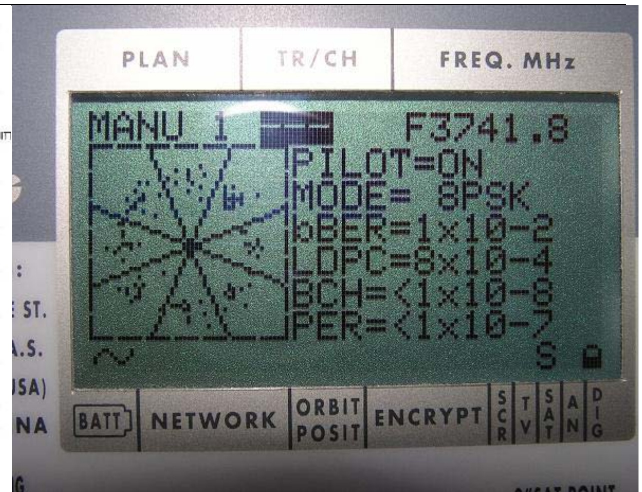


Eye pattern for PAM M=2



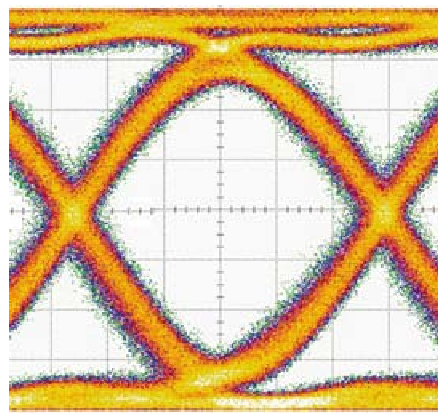
In PAM modulation with M-ary, we see M-1 eyes





Two dimensional eye pattern

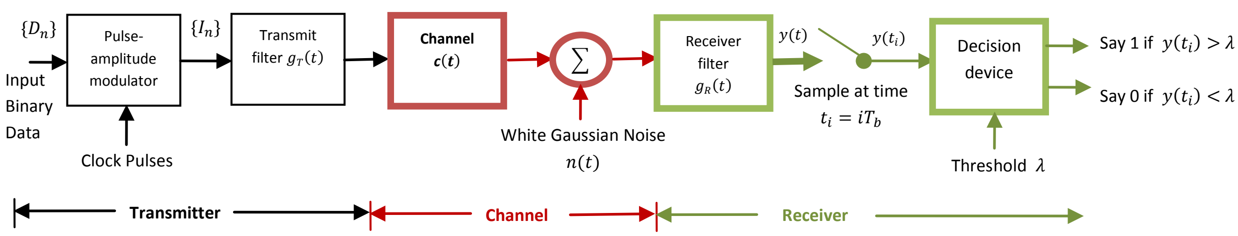
Practice Problem



Sketch the eye diagram given the following values: (Proper scale)

Distortion of zero crossing =10 μs.
Time interval over which the received signal can be sampled = 100 μs.
Margin over noise = 4 V.
Distortion at sampling time =1 V.
Approximately, what is the Sensitivity to timing error?

Distortionlesss Bandlimited Channel



$C(f) = |C(f)|e^{-j\phi(f)}$, has a constant amplitude and linear phase for $|f| \leq W$

Assuming matched filter, the output of the matched filter is

$$y(t) = \sum_{n=0}^{\infty} I_n x(t - nT) + v(t)$$

$x(t) = h(t) * g_R(t)$ and $v(t) = z(t) * g_R(t)$

The output at the sampling times $t = kT + \tau_0, k = 0,1, \ldots.$

$$y(kT + \tau_0) = \sum_{n=0}^{\infty} I_n x(kT + \tau_0 - nT) + v(kT + \tau_0)$$

where τ_0 is the transmission delay through the channel.

To simplify the notation

$$y_k = \sum_{n=0}^{\infty} I_n x_{k-n} + v_k, \quad k = 0,1,2, \ldots$$

$$= x_0 I_k + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} + v_k, \quad k = 0,1,2, \ldots$$

We may set x as 1 for simplicity

$$y_k = I_k + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} + v_k, \quad k = 0,1,2, \ldots$$

Desired term + ISI + additive noise (zero mean, variance=$\frac{N_0}{2} E_h$)

Two distortions are caused by ISI:

a) The eye closes => less noise margin => higher probability of error.
b) Distort the zero crossing => Synchronization errors.

Design of Bandlimited Signals for zero ISI-The Nyquist Criteria

To remove ISI in time domain we require

$$x(mT - nT) = 0 \; for \; m \neq n$$

$$and$$

$$x(0) \neq 0, \quad (m = n)$$

Assuming matched filter & ideal channel $C(f) = 1 \; for \; |f| < W$

The pulse $x(t)$has a spectral characteristics

$X(f) = |G(f)|^2$ $\qquad x(t) = \int_{-W}^{W} X(f) e^{j2\pi f t} df$

If we know $x(t)$, we can find $g(t)$ the transmitted pulse.

Nyquist Pulse-Shaping Criterion or Nyquist Criterion for Zero ISI

The necessary and sufficient condition for $x(t)$ to satisfy $x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

is that its Fourier transform $X(f)$ satisfy $\sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T$

Proof:

$$x(t) = \mathcal{F}^{-1}[X(f)]$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

At the sampling instances

$$x(nT) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f nT} df$$

Break the integral into finite range $1/T$ & sum

$$x(nT) = \sum_{m=-\infty}^{\infty} \int_{\frac{2m-1}{2T}}^{\frac{2m+1}{2T}} X(f) e^{j2\pi f nT} df$$

Rather than shifting the limits we are shifting the signal and fixing the limits (not perfect)

To be more precise, we do variable change $f' = f - \frac{m}{T}$ => $f = f' + \frac{m}{T}$

Integration limits, lower limit $f = \frac{m}{T} - \frac{1}{2T}$ => $f' = -\frac{1}{2T}$

Upper limit $f = \frac{m}{T} + \frac{1}{2T}$ => $f' = +\frac{1}{2T}$

Function becomes $X(f) = X\left(f' + \frac{m}{T}\right)$

And the exponent $e^{j2\pi f nT} = e^{j2\pi\left(f' + \frac{m}{T}\right)nT} = e^{j2\pi f' nT}$, because shifting to multiples of 2π has no effect.

Recalling f' as f

$$x(nT) = \sum_{m=-\infty}^{\infty} \int_{\frac{-1}{2T}}^{\frac{1}{2T}} X\left(f + \frac{m}{T}\right) e^{j2\pi f nT} df$$

$$x(nT) = \int_{-1/2T}^{\frac{1}{2T}} \left[\sum_{m=-\infty}^{\infty} X(f+\frac{m}{T})\right] e^{j2\pi fnT} df$$

$x(nT) = \int_{-\frac{1}{2T}}^{\frac{1}{2T}} B(f) e^{j2\pi fnT} df$..........(1)

where $B(f) = \left[\sum_{m=-\infty}^{\infty} X(f+\frac{m}{T})\right]$

$B(f)$ is periodic with period $1/T$. It can be represented using Fourier coefficients $\{b_n\}$ as

$B(f) = \sum_{n=-\infty}^{\infty} b_n e^{2\pi nfT}$ ....(2) where $b_n = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} B(f) e^{-j2\pi nfT} df$..........(3)

Note: we are used to do that in the time domain but it is just a dummy variable.

Comparing (1) and (3)

$b_n = Tx(-nT)$. The necessary and sufficient conditions in terms of b_n

$$b_n = \begin{cases} T & n=0 \\ 0 & n \neq 0 \end{cases}$$

Now in terms of $B(f)$ from equation (2)

$B(f) = T$

In terms of $X(f)$

$$\sum_{m=-\infty}^{\infty} X\left(f+\frac{m}{T}\right) = T$$

This is the condition for zero ISI.

Three important cases

For the ideal band limited channel, we distinguish three cases.

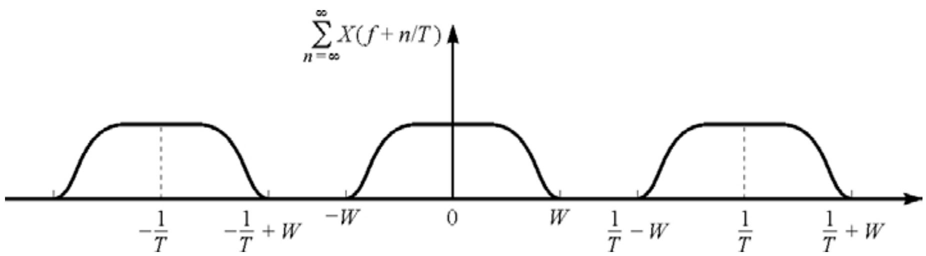
Case I $T < \frac{1}{2W}$ or $\frac{1}{T} > 2W$ (Transmission >2W)

$B(f)$ consists of non-overlapping replicas of *X*(*f*). There is non-overlapping replicas of *X*(*f*)

No chance for *X*(*f*) to make

$$\sum_{m=-\infty}^{\infty} X\left(f+\frac{m}{T}\right) = T$$

No-way to design the system for zero ISI & transmission rate>2W

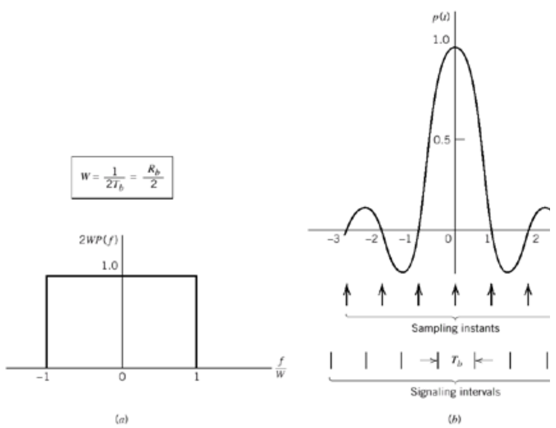


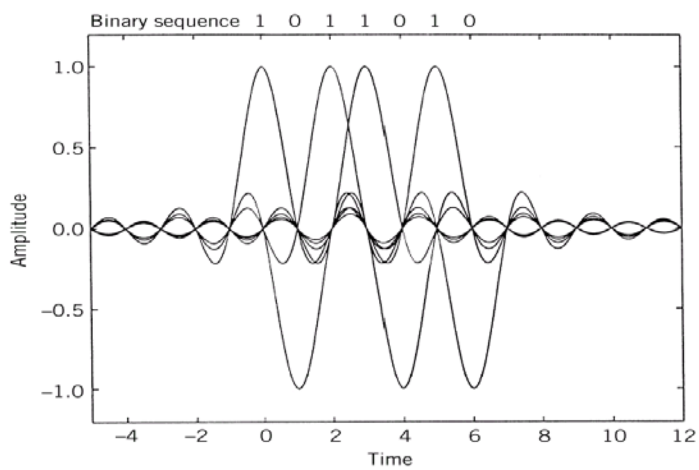
Case II $T = \frac{1}{2W}$ or $\frac{1}{T} = 2W$ (Nyquist rate)

There exists only one $X(f) = \begin{cases} T & |f| < W \\ 0 & otherwise \end{cases}$

$X(f) = T\, rect\left(\frac{f}{2W}\right) \Rightarrow x(t) = sinc\left(\frac{\pi t}{T}\right) = sinc(2W\pi t)$

Note some textbooks define $sinc(x) = \sin(\pi x)/x$ & some others $sinc(x) = \sin(x)/x$





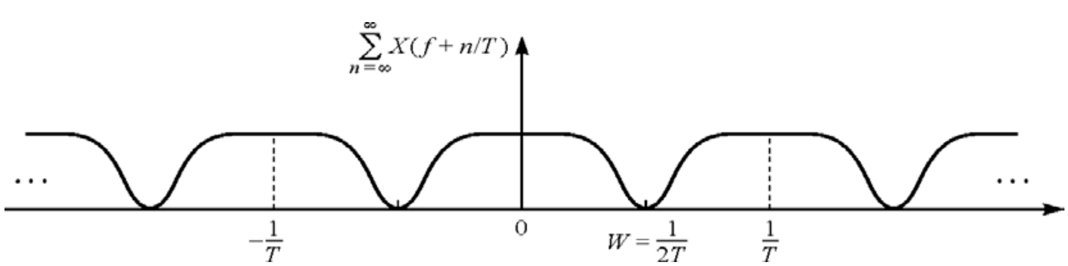
<u>Difficulty with sinc:</u>

A) non causal (non-realizable): can be reduced by using a delayed version of sinc

$sinc\left(\frac{t-t_0}{T}\right) \approx 0\ for\ t < 0$, sampling is also shifted to $mT + t_0$

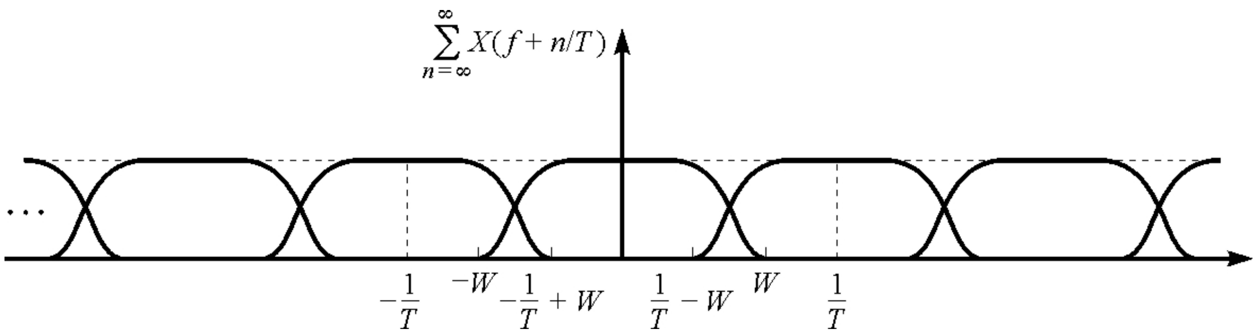
B) rate of decay (convergence) 1/t

C) Synchronization?



Case III $T > \frac{1}{2W}$

$B(f)$ Consists of overlapping replicas separated by $1/T$. We have numerous choices (symmetry)



Raised cosine (desirable spectral properties, used in practice widely)

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$

β roll-off factor $0 \le \beta \le 1$

β is proportional to excess bandwidth (bandwidth beyond $\frac{1}{2T}$) & usually is represented as percentage of the Nyquist freq

Example

$\beta = \frac{1}{2}$=> Excess bandwidth 50%

$\beta = 1$=> Excess bandwidth 100%

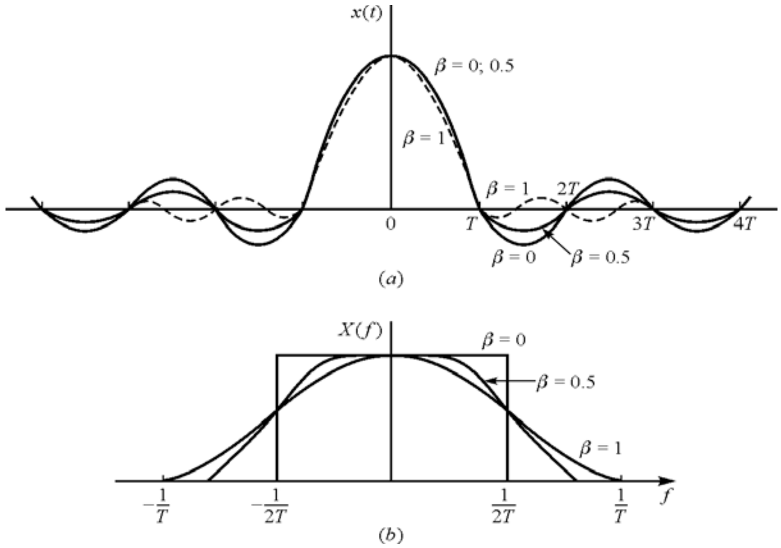
Raised cosine $x(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \frac{\cos\left(\frac{\pi\beta t}{T}\right)}{1-\frac{4\beta^2 t^2}{T^2}}$

$x(t)$ *is normalized*

$x(0) = 1$

In general, decay factor $\frac{1}{t^3}$ timing errors. Converges to a finite value. (correct the book $\beta > 0$)

Zero crossing in between.



For the ideal case $C(f) = 1$ for $|f| \leq W$

$$X_{rc}(f) = G_T(f)G_R(f)$$

$G_T(f)$: frequency response of transmit filter

$G_R(f)$: frequency response of receive filter

If the receiver filter is matched to the Tx

$$G_T(f)G_R(f) = |G_T(f)|^2$$

$$G_T(f) = \sqrt{|X_{rc}(f)|}e^{-j2\pi f t_0}$$

$$G_R(f) = G_T^*(f)$$

$$G_R(f) = \sqrt{|X_{rc}(f)|}e^{-j2\pi f (t_0+t_1)}$$

t_0 is the nominal delay to ensure realizability of the filter

The overall raised cosine spectral characteristics is split evenly between the Tx filter & the Rx filter (root raised cosine)

The bandwidth and rate relation will change to $BW = (1+\beta)R/2$ for baseband and $BW = (1+\beta)R$ for Passband.

Very important Matlab demos /Communications Toolbox

1) Eye Diagram 2) Raised Cosine filtering

Practice Example: Baseband Pulse Transmission

An analog signal is sampled, quantized, and encoded into a binary PCM wave. The number of quantization levels used is 128. A synchronizing pulse is added at the end of each code word representing a sample of the analog signal. The resulting PCM wave is transmitted over a channel of bandwidth 12 kHz using a 16-ary PAM system with raised-cosine spectrum. The roll-off factor is 0.5.

(a) Find the rate (bits/sec) at which information is transmitted through the channel.

(b) Find the rate at which the analog signal is sampled. What is the maximum possible value for the highest frequency component of the analog signal?

Design of Bandlimited Signals with Controlled ISI Partial-Response Signals

We have reduced the rate to realize practical zero ISI. Can we achieve zero ISI with a rate =2W ? The answer is yes with controlled ISI.

Duo-binary Signal:

$$x(nT) = \begin{cases} 1 & n = 0,1 \\ 0 & otherwise \end{cases}$$

$$b_n = \begin{cases} T & n = 0, n = -1 \\ 0 & otherwise \end{cases}$$

Recall $b_n = Tx(-nT)$

$$B(f) = T + Te^{-j2\pi fT}$$

Recall $B(f) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi fT} = \sum_{n=-\infty}^{\infty} X(f + \frac{m}{T})$

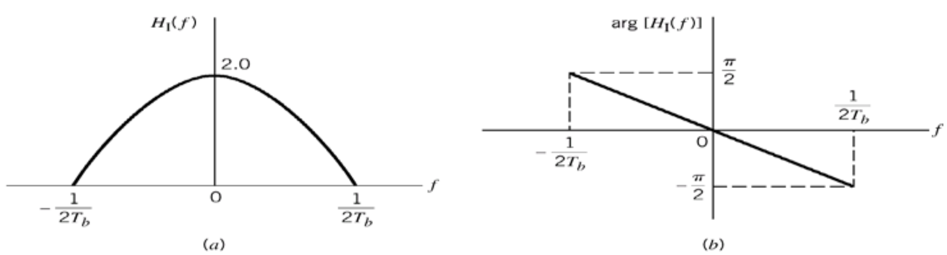
This cannot be satisfied for $T < \frac{1}{2W}$

For $T = \frac{1}{2W}$

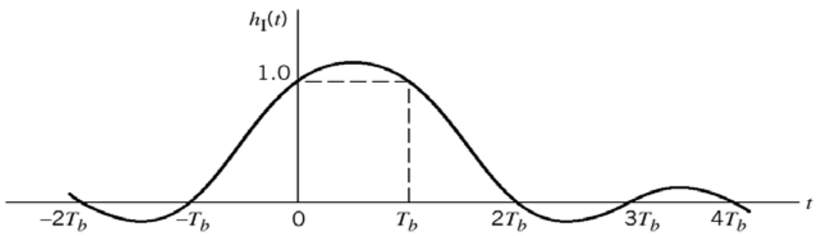
$$X(f) = \begin{cases} \frac{1}{2W}\left(1 + e^{-\frac{j\pi f}{W}}\right) & |f| < W \\ 0 & otherwise \end{cases} = \begin{cases} \frac{1}{W} e^{-\frac{j\pi f}{2W}} \cos\left(\frac{\pi f}{2W}\right) & |f| < W \\ 0 & otherwise \end{cases}$$

For which

$$x(t) = sinc\,(2\pi Wt) + sinc\left[2\pi\left(Wt - \frac{1}{2}\right)\right]$$



Frequency response of the duo-binary conversion filter. (*a*) Magnitude response. (*b*) Phase response



Impulse response of the duo-binary filter.

Modified Duo-binary Signal

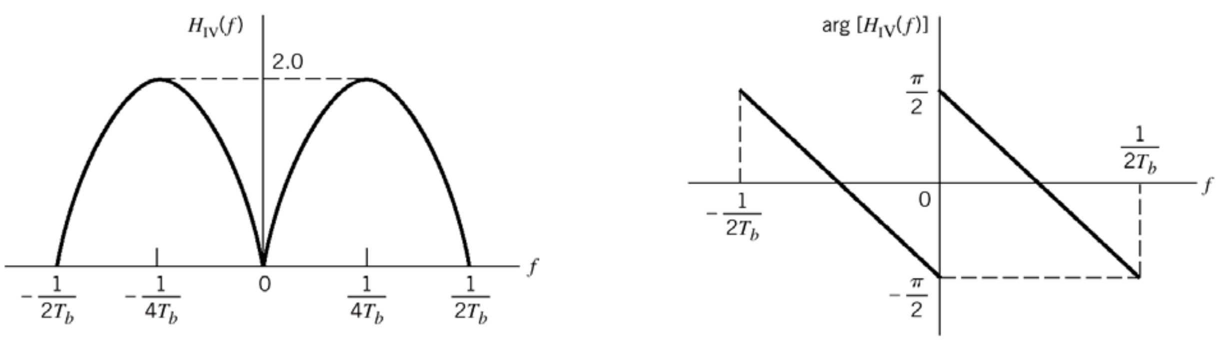
Zero DC

$$x\left(\frac{n}{2W}\right) = x(nT) = \begin{cases} 1 & n = -1 \\ -1 & n = 1 \\ 0 & otherwise \end{cases}$$

$$x(t) = sinc\left(\frac{\pi(t+T)}{T}\right) - sinc\left(\frac{\pi(t-T)}{T}\right)$$

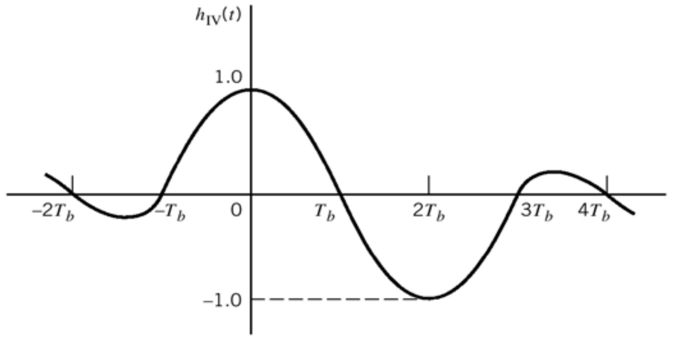
Its spectrum

$$X(f) = \begin{cases} \frac{1}{2W}\left(e^{\frac{j\pi f}{W}} - e^{\frac{-j\pi f}{W}}\right) & |f| \le W \\ 0 & |f| > W \end{cases}$$



(a) (b)

Frequency response of the modified duobinary conversion filter. (*a*) Magnitude response. (*b*) Phase response.



Impulse response of the modified duo-binary conversion filter (delayed one symbol).

In general:

- We may select different values for the other controlled ISI.
- We also may consider more samples. Why not? Difficult to unravel.

Data Detection for Controlled ISI

1) Symbol by symbol (relatively easy)
2) Maximum-likelihood criterion (minimize Probability of error (more complex)

We concentrate on PAM,

Assume $|G_T(f)| = |G_R(f)| = |X(f)|^{\frac{1}{2}}$

Symbol by symbol suboptimum detection

$$x(nT) = 1\ for\ n = 0{,}1\ \ and\ zero\ otherwise$$

At the output of the receiver filter (demodulator)

$$y_m = B_m + v_m = I_m + I_{m-1} + v_m$$

$I_m$is transmitted seq.

v_m samples of additive Gaussian noise (not white)

Let $I_m = \pm 1$ with equal probability

$$B_m = -2{,}0{,}2$$

$$Prob. = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$$

If one symbol is detected subtract its contribution from the next symbol. Error is reinforced by incorrect subtraction (Error Propagation). To avoid error propagation (Precoding) is used

$\{D_n\}$ Data sequence 0.....1

$\{P_n\}$ precoded sequence

$$P_m = D_m - P_{m-1},\ \ m = 1{,}2, \dots ..$$

$where - is\ module\ 2\ subtraction$. Like addition but more illustrative

Then $I_m = -1\ for\ P_m = 0$

$$I_m = 1\ for\ P_m = 1$$

$$I_m = 2P_m - 1$$

Noise free samples at the output of the receiver filter

$$B_m = I_m + I_{m-1}$$

$$= (2P_m - 1) + (2P_{m-1} - 1) = 2(P_m + P_{m-1} - 1)$$

$$P_m + P_{m-1} = \frac{1}{2}B_m + 1$$

$$D_m = P_m + P_{m-1} = \frac{1}{2}B_m + 1$$

Thus

$$B_m = \pm 2\ \ means\ \ \ \ D_m = 0$$

$$B_m = 0\ \ means\ \ \ \ D_m = 1$$

Easy decoding

For noisy data, we may use threshold decoding $D_m = \begin{cases} 1 & |y_m| < 1 \\ 0 & |y_m| \geq 1 \end{cases}$

D_m		1	1	1	0	1	0	0	1	0	0	0	1	1	0	1
P_m	0	1	0	1	1	0	0	0	1	1	1	1	0	1	1	0
I_m	-1	1	-1	1	1	-1	-1	-1	1	1	1	1	-1	1	1	-1
B_m		0	0	0	2	0	-2	-2	0	2	2	2	0	0	2	0
D_m		1	1	1	0	1	0	0	1	0	0	0	1	1	0	1

Extension for multilevel PAM: (duobinary)

$$B_m = I_m + I_{m-1} \dots\dots m = 1,2,\dots$$

$2M-1$ Levels for B_m

$$I_m = 2P_m - (M-1)$$

where$\{P_m\}$ is the precoded sequence obtained from the M-level data sequence.

$$P_m = D_m - P_{m-1} \dots\dots (mod\ M)$$

$\{D_m\}$ are having possible values $0,1,2,\dots.M-1$

$$D_m = 0,1,2,\dots..M-1$$

$$I_m = \pm d, \pm 3d, \dots..\pm(M-1)d$$

$$B_m = 0, \pm 2d, \pm 4d, \dots..\pm 2(M-1)d \dots\dots..noise\ free\ case$$

So we may write

$$B_m = I_m + I_{m-1} = 2[P_m + P_{m-1} - (M-1)]$$

Since $D_m = P_m + P_{m-1} \quad (mod\ M)$

$$D_m = \frac{1}{2}B_m + (M-1) \quad (mod\ M)$$

The following is an in class example:

D_m		0	0	1	3	1	2	0	3	3	2	0	1	0
P_m	0	0	0	1	2	3	3	1	2	1	1	3	2	2
I_m	-3	-3	-3	-1	+1	+3	+3	-1	+1	-1	-1	+3	+1	+1
B_m		-6	-6	-4	0	4	6	2	0	0	-2	2	4	2
D_m		0	0	1	3	1	2	0	3	3	2	0	1	0

- Note that there is some error correction capability (check Lathi's text book)

- In case of noise, we have to quantize to the nearest level

For modified duobinary:

$$x\left(\frac{n}{2W}\right) = -1 \quad for\ n = 1$$

$$x\left(\frac{n}{2W}\right) = 1\ for\ n = -1$$

$$b_m = I_m - I_{m-2}$$

$$P_m = D_m + P_{m-2} \quad (mod\ M)$$

Conclusion about symbol by symbol detection:

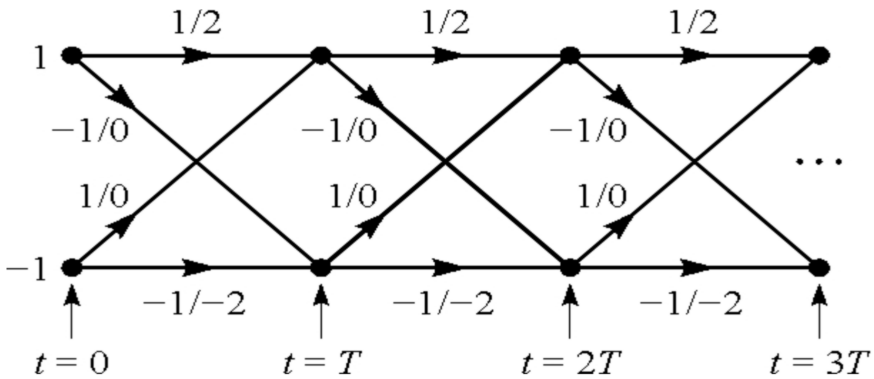
+ Simple and used practically in many systems employing duobinary and modified duobinary.

- Mot optimal (in case of noise) due to the inherent memory in the received signal of a system using partial response signaling. This memory is not utilized.

This motivates the subject of MLS detection which utilizes the inherent memory.

Maximum Likelihood Sequence Detection

Memory can be represented by a trellis. For Duobinary signaling the trellis is shown in the Figure



- Duobinary has a memory of length $L = 1$.
- The trellis has $s_t = M^L = 2^1\ states$
- Optimum ML: select the most probable path through the trellis upon observing $\{y_m\}\ at\ t = mT, \ldots m = 1,2, \ldots.$
- Every node will have Mincoming nodes with corresponding metrics.

Viterbi Algorithm

- To learn more about Viterbi you might visit Dr. Muqaibel's Website for Digital Communications 1
- Out of the Mchoose one and discarded $M - 1$.
- Surviving path will be extended to M paths

For a class of partial response signals the received sequence $\{y_m, 1 \leq m \leq N\}$ is described by the multivariate Gaussian pdf

$$\underline{y_N} = \{y_1\ y_2\ y_3\ \ \ldots.\ y_N\}'$$

$$\underline{I_N} = \{I_1\ I_2\ I_3\ \ \ldots.\ I_N\}' \qquad ,N > L$$

With additive noise which Gaussian with zero mean

$$P\left(\underline{y_N}\middle|\underline{I_N}\right) = \frac{1}{(2\pi\ det\ \underline{C})^{\frac{N}{2}}} \exp\left(-\frac{1}{2}\left(\underline{y_N} - \underline{B_N}\right)' \underline{C}^{-1}\left(\underline{y_N} - \underline{B_N}\right)\right)$$

Where $\underline{B_N} = [\ B_1\ \ B_2\ \ \ B_3\ \ \ \ldots \ldots B_N]'$ is the mean of the vector y_N

$\underline{C}$ is the $N \times N$covariance matrix of $\underline{y_N}$

The ML selects the sequence that maximizes the pdf or maximize the natural logarithm (monotonic function) of $P\left(\underline{y_N}\middle|\underline{I_N}\right)$

$$\ln P\left(\underline{y_N}\middle|\underline{I_N}\right) = -\frac{N}{2}\ln(2\pi\, det\, \underline{C}) - \frac{1}{2}\left(\underline{y_N} - \underline{B_N}\right)' \underline{C}^{-1}\left(\underline{y_N} - \underline{B_N}\right)$$

Given $\{y_m\}$ the data sequence I_m that maximize $\ln P\left(\underline{y_N}\middle|\underline{I_N}\right)$ is identical to the sequence I_N that minimize $\left(\underline{y_N} - \underline{B_N}\right)' \underline{C}^{-1}\left(\underline{y_N} - \underline{B_N}\right)$

$$\widehat{I_N} = \arg_{I_N} \min\left[\underline{\left(\underline{y_N} - \underline{B_N}\right)' C^{-1}}\left(\underline{y_N} - \underline{B_N}\right)\right]$$

The computation is complicated by the calculation of the correlation of the noise samples v_m at the output of the matched filter.

$$E[v_m v_{m+k}] = 0 \quad for\ k > L$$

Simplification:

$$E[v_m v_{m+k}] = 0 \quad for\ k > 0$$

This means that $\underline{C} = \sigma_v^2 \underline{1_N}$, where $\sigma_v^2 = E[v_m^2]$,
$\underline{1_N} = N \times N\ identity\ matrix\ ..avoid\ confusion\ with\ \underline{I_N}$

$$\widehat{I_N} = \arg_{I_N} \min\left[\left(\underline{y_N} - \underline{B_N}\right)'\left(\underline{y_N} - \underline{B_N}\right)\right]$$

$$\widehat{I_N} = \arg_{I_N} \min\left[\sum_{m=1}^{N}\left(y_m - \sum_{k=0}^{L} x_k I_{m-k}\right)^2\right]$$

where

$$B_m = \sum_{k=0}^{L} x_k I_{m-k}$$

$x_k = x(kT)$sample values of the practical responses waveform

Metric Computation:

$$DM_m\left(\underline{I_m}\right) = DM_{m-1}\left(\underline{I_{m-1}}\right) + \left(y_m - \sum_{k=0}^{L} x_k I_{m-k}\right)^2$$

$$\left(y_m - \sum_{k=0}^{L} x_k I_{m-k}\right)^2 \ new\ metric\ based\ on\ the\ received\ y_m$$

$$DM_m\left(\underline{I_m}\right)\ distance\ metric\ at\ t = mT$$

$$DM_{m-1}\left(\underline{I_{m-1}}\right) \ distance\ metric\ at\ t = (m-1)T$$

Problem: ML: introduces variable delay in detecting each transmitted symbol.

Solution: truncate to N_t most recent symbols. $N_t > 5L$ if still disagree choose the most probable path. Loss in performance after $5L$ is negligible.

Example (MLS)

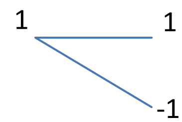
Let us say that we transmitted $I_0\ \ I_1\ \ I_2\ \ I_3 \ldots. = 1\ \ 1\ \ \ 1\ \ 1\ \ 1 \ldots..$

The received nose free sequence is $b_1\ \ b_2\ \ b_3\ \ \ b_4 \ldots. = 2\ \ 2\ \ \ 2\ \ 2 \ldots..$

Due to noise the received sequence is $y_1\ \ y_2\ \ y_3\ \ \ y_4 \ldots. = 2.1\ \ 0.9\ \ \ 1.8\ \ 2.2 \ldots..$

$DM_0(1) = 0\ (Assume\ the\ first\ metric\ is\ known\ so\ the\ error\ metric\ is\ zero)$

$DM_1(1,1) = 0 + (y_1 - 2)^2 = (2.1 - 2)^2 = 0.01$



$$DM_1(1,-1) = 0 + (y_1 - 0)^2 = (2.1)^2 = 4.41$$

$$\boldsymbol{DM_2(1,1,1) = 0.01 + (y_2 - 2)^2 = 0.01 + 1.21 = 1.22\ (Survivor)}$$

$$DM_2(1,-1,1) = 4.41 + (y_2 - 0)^2 = 4.41 + 0.81 = 5.22$$

$$\boldsymbol{DM_2(1,1,-1) = 0.01 + (y_2 - 0)^2 = 0.01 + 0.81 = 0.82\ (Survivor)}$$

$$DM_2(1,-1,-1) = 4.41 + (y_2 + 2)^2 = 4.41 + 2.9^2 = \cdots$$

$$(no\ need\ for\ calculation\ becuase\ it\ clearly\ not\ the\ survivor$$

$$\boldsymbol{DM_3(1,1,1,1) = 1.22 + (y_3 - 2)^2 = 1.22 + 0.04 = 1.26\ (Survivor)}$$

$$DM_3(1,1,-1,1) = 0.82 + (y_3 - 0)^2 = 0.82 + 1.8^2 = 4.06$$

$$\boldsymbol{DM_3(1,1,1,-1) = 1.22 + (y_3 - 0)^2 = 1.22 + 3.24 = 4.46\ (Survivor)}$$

$$DM_3(1,1,-1,-1) = 0.82 + (y_3 - 0)^2 = 0.82 + 3.8^2 =$$

ML output is $\quad I_1 = \mathbf{1}, I_2 = \mathbf{1}, I_3 = \mathbf{1}$

Using Symbol by Symbol

$$I_m = y_m - I_{m-1}$$

$$I_1 = 1.1 \ldots \ldots .1$$

$$I_2 = 0.9 - 1 = -0.1 \ldots. -1\ (not\ correct)$$

$I_3 = 1.8 - (-1) = 2.8 \ldots \ldots \ldots 1$

Signal Design for Channels with Distortion

1) Bandlimited Ideal Channels was done **(Before)**

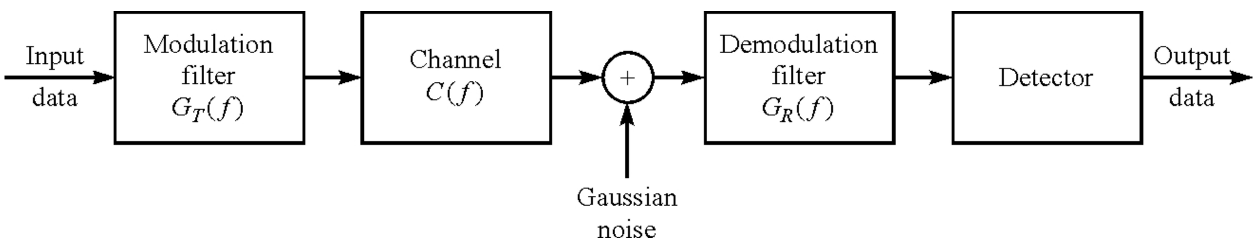
2) Channels with distortions (Now) (time-invariant) **(Now)**

3) Use Equalization to mitigate time variant distortion **(Later)**

$$C(f) \text{is known } |f| \leq W, \quad C(f) = 0 \text{ for } |f| > W$$

Objective: find $G_T(f)$ and $G_R(f)$ to minimize the error probability at the detector.

Noise is assumed to be Gaussian with PSD $S_{nn}(f)$



We must satisfy

$$G_T(f)C(f)G_R(f) = X_d(f)e^{-j2\pi f t_0} \qquad , |f| \leq W$$

$X_d(f)$ is chosen to get zero ISI or controlled ISI.

For zero ISI, let $X_d(f) = X_{rc}(f)$ raised cosine

Noise $v(t) = \int_{-\infty}^{\infty} n(t-\tau) g_R(\tau) d\tau$

$$S_{vv}(f) = S_{nn}(f)|G_R(f)|^2$$

$$\sigma_v^2 = \int_{-\infty}^{\infty} S_{nn}(f)|G_R(f)|^2 df$$

Consider PAM for simplicity $y_m = x_0 I_m + v_m = I_m + v_m$ where x_0 is normalized $I_m = \pm d$

$$P_2 = \frac{1}{\sqrt{2\pi}} \int_{d/\sigma_v}^{\infty} e^{-\frac{y^2}{2}} dy = Q\left(\sqrt{\frac{d^2}{\sigma_v^2}}\right)$$

We need to maximize $\frac{d^2}{\sigma_v^2}$. Assuming noise is white $S_{nn}(f) = \frac{N_0}{2}$

Scenario #1: Pre-compensation for the total Channel distortion at the transmitter

$$|G_T(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|} \qquad |f| \leq W$$

$|G_R(f)| = \sqrt{X_{rc}(f))} \quad |f| \leq W$

Phase is also compensated at the transmitter.

$$P_{av} = \frac{E(I_m^2)}{T}\int_{-\infty}^{\infty} g_T^2(t)dt = \frac{d^2}{T}\int_{-W}^{W} |G_T(f)|^2 df$$

$$= \frac{d^2}{T}\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|^2} df$$

$$d^2 = P_{av}T\left[\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|^2} df\right]^{-1}$$

The noise at the output of the receiver $\sigma_v^2 = \frac{N_0}{2}$,, Hence SNR

$$\frac{d^2}{\sigma_v^2} = \frac{2P_{av}T}{N_0}\left[\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|^2} df\right]^{-1}$$

Scenario #2: Compensation is equally divided between the transmitter and the receiver filters

$|G_T(f)| = |G_R(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{\frac{1}{2}}} \quad |f| \leq W$, the phase is also divided

$$P_{av} = \frac{d^2}{T}\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df$$

Noise variance $\sigma_v^2 = \frac{N_0}{2}\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df$

The SNR is given by

$$\frac{d^2}{\sigma_v^2} = \frac{2P_{av}T}{N_0}\left[\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df\right]^{-2}$$

Loss (dB) due to scenario #1 $\quad 10log\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|^2} df$

Loss (dB) due to scenario #2 $\quad 10log\left[\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df\right]^{-2}$

If the channel is ideal $C(f) = 1, |f| < W$, which means $\int_{-W}^{W} X_{rc}(f)df = 1$. (No loss)

If there is loss ($|C(f)| < 1) \ |f| \leq W$), scenario # 2 might be better. (See text #9.30)

CDMA IS-95 uses root raised cosine with roll-off factor 0.22

Example

For a binary transmission system with rate =4800 bits/sec over a channel with frequency (magnitude) response $|C(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{W}\right)^2}}, \ |f| \le W$

Where W=4800Hz . Noise is AWGN with zero mean and $\frac{N_0}{2} = 10^{-15}\frac{W}{Hz}$

$\beta = 1$, because $\frac{1}{T} = W$

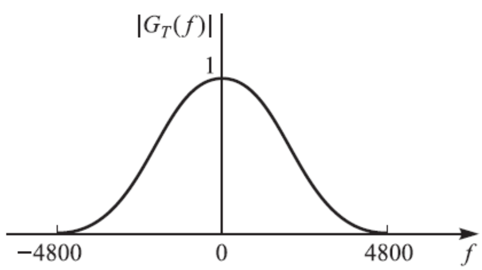
$$X_{rc}(f) = \frac{1}{2}T[1 + cos(\pi T/f)] = Tcos^2\left(\frac{\pi|f|}{9600}\right)$$

Using equal compensation

$$|G_T(f)| = |G_R(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{\frac{1}{2}}} = \left[1 + \left(\frac{f}{4800}\right)^2\right]^{\frac{1}{4}} cos\left(\frac{\pi|f|}{9600}\right) \quad |f| \le 4800$$

and zero otherwise

One can find the amount fo energy required to achieve a certain probability of error or loss when using the other compensation scenario.



Frequency response of an optimum transmitter filter.

Probability of Error in detection of PAM

M-ary in AWGN

Consider 1) $G_T(f)$ & $G_R(f)$ are designed for zero ISI.

2) $x(t) = g_T(t) * g_R(t)$ duobinary or modified duobinary.

Probability of Error Detection for PAM with Zero ISI

If no ISI $y_m = x_0 I_m + v_m$

$$x_0 = \int_{-W}^{W} |G(f)|^2 df = E_g$$

v_m is zero mean & $\sigma_v^2 = \frac{1}{2} E_g N_0$

I_m takes one out of M possible values.

Objective: P_e for a given amplitude.

Given $P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_g}{N}}\right)$

$E_g = 3\frac{E_{av}}{M^2-1}$,

$E_{symbol\_av} = kE_{bit\_av}$, $\quad k = \log_2 M$

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\, E_g}{(M^2-1)N}}\right)$$

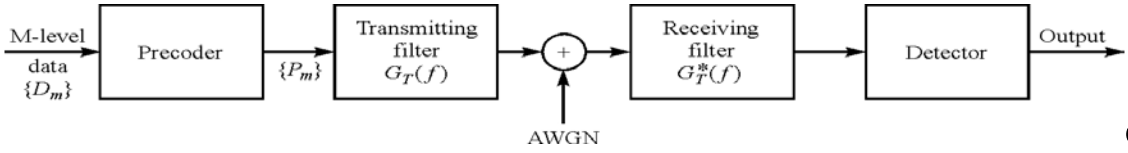
Conclusion : No change in probability of error if

- Signals are designed for zero ISI.
- Channel does not distort the transmitted signal.
- Optimum matched filter.

Probability of Error Partial Response signals

Ideal channel.. AWGN

Consider duobinary, modified duobinary.



Consider symbol-by-symbol, optimum ML sequence detection.

Pe in Symbol-by-symbol detection for controlled ISI

For the duo-binary signal:

$$y_m = I_m + I_{m-1} + v_m = B_m + v_m$$

For modified duobinary

$$y_m = I_m - I_{m-2} + v_m = B_m + v_m$$

$I_m = \pm d, \pm 3d, \ldots\ldots \pm (M-1)d$ equi-probable

$B_m = 0, \pm 2d, \pm 4d, \ldots\ldots, \pm 2(M-1)d$ total $2M-1$ levels

The scale factor d is equivalent to $x_0 = E_g$

In the absence of noise the received level has a triangular probability (show)

$$P(B = 2md) = \frac{M - |m|}{M^2} \quad , \qquad m = 0, \pm 1, \pm 2, \ldots\ldots, \pm (M-1)$$

Assume AWGN with zero mean and $PSD = \frac{1}{2} N_0$

Assume errors occur if noise exceeds d (ignore the case where that could lead to a correct decision (edges, maximum and min levels))

$$\sigma_v^2 = \frac{1}{2} N_0 \int_{-W}^{W} |G_R(f)|^2 df = \frac{1}{2} N_0 \int_{-W}^{W} |X(f)| df = \frac{\frac{N_0}{2}}{\pi/4} = \frac{2N_0}{\pi}$$

For both duobinary and modified duobinary.

An upper bound on the symbol probability of error. (See text for details)

$$P_M < 2\left(1 - \frac{1}{M^2}\right) Q\left(\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{(M^2-1)} \frac{E_{av}}{N_0}}\right)$$

E_{av} is the average energy per transmitted symbol, $E_{av} = kE_{bav} = (\log_2 M)E_{bav}$

The above is true for both duobinary and modified duobinary.

Compared with zero ISI there is a loss of 2.1 dB or $\left(\frac{1}{4}\pi\right)^2$.

The reason for the loss is the use of the suboptimal symbol-by-symbol detection.

Pe in Max-Likelihood sequence detection for controlled ISI

The loss is completely recovered. (If interested in the proof, please see the course textbook)