CHAPTER 14

Problem 14.1 :

Based on the info about the scattering function we know that the multipath spread is $T_m = 1\ ms$, and the Doppler spread is $B_d = 0.2\ Hz$.

(a) (i) $\mathrm{T}_m = 10^{-3}$ sec
(ii) $B_d = 0.2\ Hz$
(iii) $(\Delta t)_c \approx \frac{1}{B_d} = 5$ sec
(iv) $(\Delta f)_c \approx \frac{1}{T_m} = 1000\ Hz$
(v) $\mathrm{T}_m B_d = 2 \cdot 10^{-4}$

(b) (i) Frequency non-selective channel : This means that the signal transmitted over the channel has a bandwidth less that 1000 Hz.
(ii) Slowly fading channel : the signaling interval T is $T << (\Delta t)_c$.
(iii) The channel is frequency selective : the signal transmitted over the channel has a bandwidth greater than 1000 Hz.

(c) The signal design problem does not have a unique solution. We should use orthogonal M=4 FSK with a symbol rate of 50 symbols/sec. Hence $T = 1/50$ sec. For signal orthogonality, we select the frequencies with relative separation $\Delta f = 1/T = 50$ Hz. With this separation we obtain 10000/50=200 frequencies. Since four frequencies are requires to transmit 2 bits, we have up to $50^{th}-$order diversity available. We may use simple repetition-type diversity or a more efficient block or convolutional code of rate $\geq 1/50$. The demodulator may use square-law combining.

Problem 14.2 :

(a)

$$P_{2h} = p^3 + 3p^2(1-p)$$

where $p = \frac{1}{2+\bar{\gamma}_c}$, and $\bar{\gamma}_c$ is the received SNR/cell.

(b) For $\bar{\gamma}_c = 100,\ P_{2h} \approx 10^{-6} + 3 \cdot 10^{-4} \approx 3 \cdot 10^{-4}$
For $\bar{\gamma}_c = 1000,\ P_{2h} \approx 10^{-9} + 3 \cdot 10^{-6} \approx 3 \cdot 10^{-6}$

(c) Since $\bar{\gamma}_c >> 1$, we may use the approximation : $P_{2s} \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c}\right)^L$, where L is the order of diversity. For L=3, we have :

$$P_{2s} \approx \frac{10}{\bar{\gamma}_c^3} \Rightarrow \left\{ \begin{array}{ll} P_{2s} \approx 10^{-5}, & \bar{\gamma}_c = 100 \\ P_{2s} \approx 10^{-8}, & \bar{\gamma}_c = 1000 \end{array} \right\}$$

(d) For hard-decision decoding :

$$P_{2h} = \sum_{k=\frac{L+1}{2}}^{L} \binom{L}{k} p^k (1-p)^{L-k} \leq [4p(1-p)]^{L/2}$$

where the latter is the Chernoff bound, L is odd, and $p = \frac{1}{2+\bar{\gamma}_c}$. For soft-decision decoding :

$$P_{2s} \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c}\right)^L$$

Problem 14.3 :

(a) For a fixed channel, the probability of error is : $P_e(a) = Q\left(\sqrt{\frac{a^2 2\mathcal{E}}{N_0}}\right)$. We now average this conditional error probability over the possible values of α, which are a=0, with probability 0.1, and a=2 with probability 0.9. Thus :

$$P_e = 0.1 Q(0) + 0.9 Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) = 0.05 + 0.9 Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right)$$

(b) As $\frac{\mathcal{E}}{N_0} \to \infty, \ P_e \to 0.05$

(c) When the channel gains a_1, a_2 are fixed, the probability of error is :

$$P_e(a_1, a_2) = Q\left(\sqrt{\frac{(a_1^2 + a_2^2)\, 2\mathcal{E}}{N_0}}\right)$$

Averaging over the probability density function $p(a_1, a_2) = p(a_1) \cdot p(a_2)$, we obtain the average probability of error :

$$\begin{array}{rl} P_e = & (0.1)^2 Q(0) + 2 \cdot 0.9 \cdot 0.1 \cdot Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + (0.9)^2 Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \\ = & 0.005 + 0.18 Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + 0.81 Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \end{array}$$

(d) As $\frac{\mathcal{E}}{N_0} \to \infty, \ P_e \to 0.005$

Problem 14.4 :

(a)

$$T_m = 1 \text{ sec} \Rightarrow (\Delta f)_c \approx \frac{1}{T_m} = 1\ Hz$$
$$B_d = 0.01\ Hz \Rightarrow (\Delta t)_c \approx \frac{1}{B_d} = 100 \text{ sec}$$

(b) Since $W = 5\ Hz$ and $(\Delta f)_c \approx 1\ Hz$, the channel is frequency selective.

(c) Since T=10 sec $<\ (\Delta t)_c$, the channel is slowly fading.

(d) The desired data rate is not specified in this problem, and must be assumed. Note that with a pulse duration of $T = 10$ sec, the binary PSK signals can be spaced at $1/T = 0.1\ Hz$ apart. With a bandwidth of W=5 Hz, we can form 50 subchannels or carrier frequencies. On the other hand, the amount of diversity available in the channel is $W/(\Delta f)_c = 5$. Suppose the desired data rate is 1 bit/sec. Then, ten adjacent carriers can be used to transmit the data in parallel and the information is repeated five times using the total number of 50 subcarriers to achieve 5-th order diversity. A subcarrier separation of 1 Hz is maintained to achieve independent fading of subcarriers carrying the same information.

(e) We use the approximation :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{4\bar{\gamma}_c}\right)^L$$

where L=5. For $P_3 = 10^{-6}$, the SNR required is :

$$(126) \left(\frac{1}{4\bar{\gamma}_c}\right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 10.4\ (10.1\ dB)$$

(f) The tap spacing between adjacent taps is 1/5=0.2 seconds. the total multipath spread is $T_m = 1$ sec. Hence, we employ a RAKE receiver with at least 5 taps.

(g) Since the fading is slow relative to the pulse duration, in principle we can employ a coherent receiver with pre-detection combining.

(h) For an error rate of 10^{-6}, we have :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c}\right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 41.6\ (16.1\ dB)$$

Problem 14.11 :

The radio signal propagates at the speed of light, $c = 3\times10^8 m/\sec$. The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3\times10^8} = 1\mu\sec$$

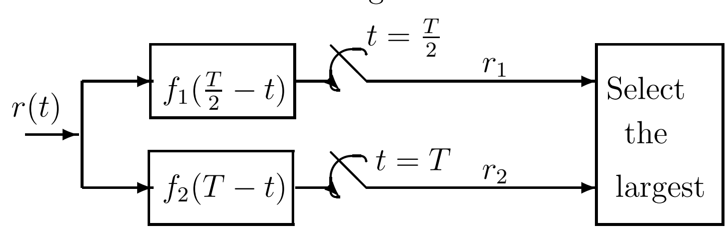
The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is $W = 1\ MHz$. Hence, the minimum chip rate is 10^6 chips per second.

Problem 14.12 :

(a) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \qquad f_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \le t < T \\ 0, & \text{otherwise} \end{cases}$$

(b) The optimal receiver is shown in the next figure



(c) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\mathbf{r} = [\sqrt{\frac{A^2T}{2}} + n_1, n_2]$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(e|s_1)$ is

$$\begin{aligned} P(e|s_1) &= P(n_2 - n_1 > \sqrt{\frac{A^2T}{2}}) \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_{\frac{A^2T}{2}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

where we have used the fact the $n = n_2 - n_1$ is a zero-mean Gaussian random variable with variance N_0. Similarly we find that $P(e|s_1) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$, so that

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$$

(d) The signal waveform $f_1(\frac{T}{2}-t)$ matched to $f_1(t)$ is exactly the same with the signal waveform $f_2(T-t)$ matched to $f_2(t)$. That is,

$$f_1(\frac{T}{2}-t) = f_2(T-t) = f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce the random variables r_1 and r_2 respectively.

(e) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_1 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_1 \\ r_2 &= \frac{A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 = \frac{1}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} V$$

then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= n_1 \\ r_2 &= A\sqrt{\frac{2}{T}}\frac{T}{4} + \frac{3A}{2}\sqrt{\frac{2}{T}}\frac{T}{4} + n_2 \\ &= \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$$

Thus, the average probability of error is given by

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= \frac{1}{2}Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right] \end{aligned}$$

The optimal value of V can be found by setting $\frac{\vartheta P(e)}{\vartheta V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta V} = 0 = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$$

or by solving in terms of V

$$V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

(f) Let a be fixed to some value between 0 and 1. Then, if we argue as in part (e) we obtain

$$\begin{aligned} P(e|s_1, a) &= P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V(a)) \\ P(e|s_2, a) &= P(n_1 - n_2 > (a+2)\sqrt{\frac{A^2T}{8}} + V(a)) \end{aligned}$$

and the probability of error is

$$P(e|a) = \frac{1}{2}P(e|s_1, a) + \frac{1}{2}P(e|s_2, a)$$

For a given a, the optimal value of $V(a)$ is found by setting $\frac{\vartheta P(e|a)}{\vartheta V(a)}$ equal to zero. By doing so we find that

$$V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$$

The mean square estimation of $V(a)$ is

$$V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}}\int_0^1 ada = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$