Spectral of digitally modulated signals

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Introduction

- **Motivation:** The constraints imposed by the channel bandwidth in the selection of the modulation technique used to transmit the information
- Information is stochastic, and hence digitally modulated signals are *stochastic processes.* =>Power Spectrum Density (PSD) (Not FFT)
- Types of digitally modulated signals
  - **Linearly modulated signals (ASK, PSK, QAM)**
  - Non linearly modulated signals (CPFSK, CPM) beyond the scope
  - Baseband with memory (Markov structure)

Power Spectra of Linearly Modulated Signals (ASK, PSK, QAM)

$\{I_n\} \rightarrow$ Represents the sequence of symbols that results from mapping k bits blocks.

Modulation scheme	$\{I_n\}$
PAM	Real value
PSK, QAM, combined (PAM-PSK)	Complex value

- $$v_l(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$$

  ↑ Low pass

  Transmitted rate

  $$\frac{1}{T} = \frac{R}{k}\ bits/sec$$

$$\because v(t) = Re\left[v_l(t)e^{j2\pi f_c t}\right]$$

$$\Rightarrow R_v(\tau) = Re\left[R_{v_l}e^{j2\pi f_c t}\right]$$

$$\because S_v = \frac{1}{2}\left[S_{v_l}(f - f_c) + S_{v_l}(-f - f_c)\right]$$

- $R_{v_l}(t+\tau;t) = \frac{1}{2}E[v_l{}^*(t)\, v_l(t+\tau)]$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[\, I_n^* I_m \,]\, g^*(t-nT)g(t+\tau-mT)$$

$\because \{I_n\}\ w.s.s$ with mean $= \mu_i$ and Autocorrelation $= R_I(m) = \frac{1}{2}E[I_n^* I_{n+m}]$

$$\Rightarrow R_{v_l}(t+\tau;t) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_I(m-n)g^*(t-nT)g(t+\tau-mT)$$

Let m'= m+n

$$R_{v_l}(t+\tau;t) = \sum_{m\prime=-\infty}^{\infty} R_I(m') \sum_{n=-\infty}^{\infty} g^*(t-nT)g(t+\tau-(m'+n)T)$$

Let m=m'

$$= \sum_{m=-\infty}^{\infty} R_I(m) \sum_{n=-\infty}^{\infty} \boxed{g^*(t-nT)g(t+\tau-nT-mT)}$$

Is periodic with period T
$\Rightarrow R_{v_l}$ is periodic of T i.e
$R_{v_l}(t+\tau;t) = R_{v_l}(t+T+\tau;t+T)$

∵The mean value of $v_l(t)$ is periodic

$$E[v_l(t)] = \mu_i \sum_{n=-\infty}^{\infty} g(t - nT)$$

- **Cyclostationary process:** periodically stationary process is w.s.

- To avoid time dependence we average over one period

$$\bar{R}_{v_l} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{v_l}(t + \tau; t) dt$$

$$= \sum_{m=-\infty}^{\infty} R_I(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g^*(t - mT) g(t + \tau - nT - mT) dt$$

Time shift by $-nT$: change of variable $n+m$ to m , n to 0 , and integration limits

$$= \sum_{m=-\infty}^{\infty} R_I(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-\frac{T}{2}-nT}^{\frac{T}{2}-nT} g^*(t - mT) g(t + \tau - mT) dt$$

- $\because$ *the Time autocorrelation function of* $g(t)$ *is*

$$R_g = \int_{-\infty}^{\infty} g^*(t)g(t+\tau)dt$$

$$\Rightarrow \bar{R}_v(\tau) = \frac{1}{T}\sum_{-\infty}^{\infty} R_I(m)R_g(\tau - mT)$$

By F.T, the average PSD

$$S_{v_l}(f) = \frac{1}{T}|G(f)|^2 S_I(f)$$

- $G(f)$ is the F.T of $g(t)$
- $S_I(f)$ is the P.S.D of the information sequence

- $$S_I(f) = \sum_{m=-\infty}^{\infty} R_I(m) e^{-j2\pi fmT}$$

Exponential Fourier series
Discrete F.T

i.e. PSD of $v(t)$ depends on

1) pulse shape g(t)

2) correlation characteristic of information sequence.

$$R_I(m) = T \int_{\frac{-1}{2T}}^{\frac{1}{2T}} S_I(f) e^{j2\pi fmT} \, df$$

- Example, $\{I_m\}$ is real and mutually uncorrelated

$\sigma_i^2 = E[I^2] - \mu_i^2$ and hence $E[I^2] = \mu_i^2 + \sigma_i^2$

$E[I_i I_{i+m}] = E[I_{i+m}] E[I_{i+m}] = \mu_i^2$

$$R_I(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m = 0) \\ \mu_i^2 & (m \neq 0) \end{cases}$$

$$S_I(f) = \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$

Periodic with period $1/T$

- It can be viewed as the exponential F.S of a periodic train of impulse with each impulse having an area of $\frac{1}{T}$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

where $\omega_s = \frac{2\pi}{T_s}$

$$\therefore S_I = \sigma_i^2 + \frac{\mu_i^2}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right)$$

- For uncorrelated sequence (suing the property of the impulse function)

$$S_{v_l}(f) = \frac{\sigma_i^2}{T} |G(f)|^2 + \frac{\mu_i^2}{T^2} \sum_{m=-\infty}^{\infty} \left|G\left(\frac{m}{T}\right)\right|^2 \delta(f - \frac{m}{T})$$

- First term: Continuous spectrum
- Second Term: Discrete frequency component spaced by $\frac{1}{T}$
- If the mean $\mu = 0 \Rightarrow$ no spectral lies $\rightarrow$ (desirable)
  - To get zero mean we need:
    - Equally likely symbols
    - Symmetrically positioned

Example I

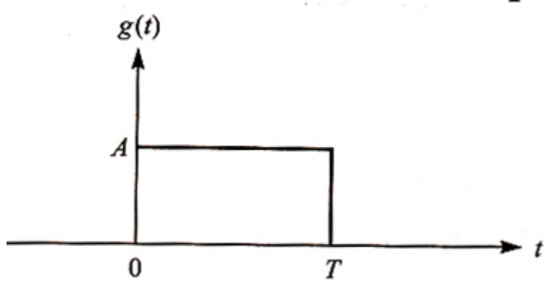
$$+1, +3, -1, -3$$

- $means = \frac{+1+3-1-3}{4} = 0$
- $variance = \frac{2(3-0)^2+2(-1-0)^2}{4} = \frac{18+2}{4} = 5$

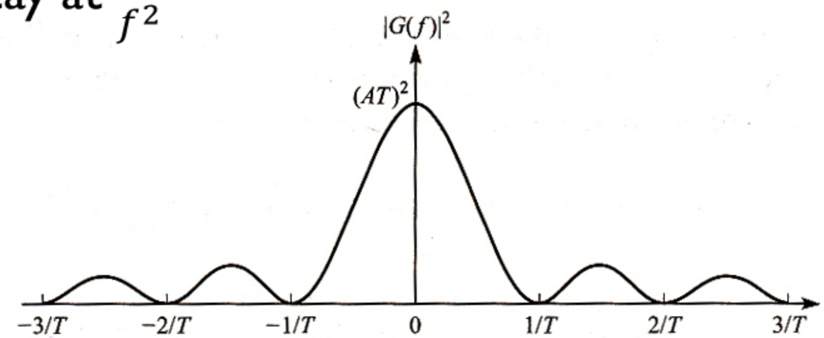
Example II To illustrate the effect of g(t)

$$G(f) = AT\frac{\sin \pi f T}{\pi f T}e^{-j\pi f T} \Rightarrow |G(f)|^2 = (AT)^2\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

- Zeros at multiple of $\frac{1}{\mathrm{T}}$ Decay at $\frac{1}{f^2}$



(a)



(b)

$$S_v(f) = \sigma_i^2 A^2 T\left(\frac{\sin \pi f T}{\pi f T}\right)^2 + A^2\mu_i^2\delta(f)$$

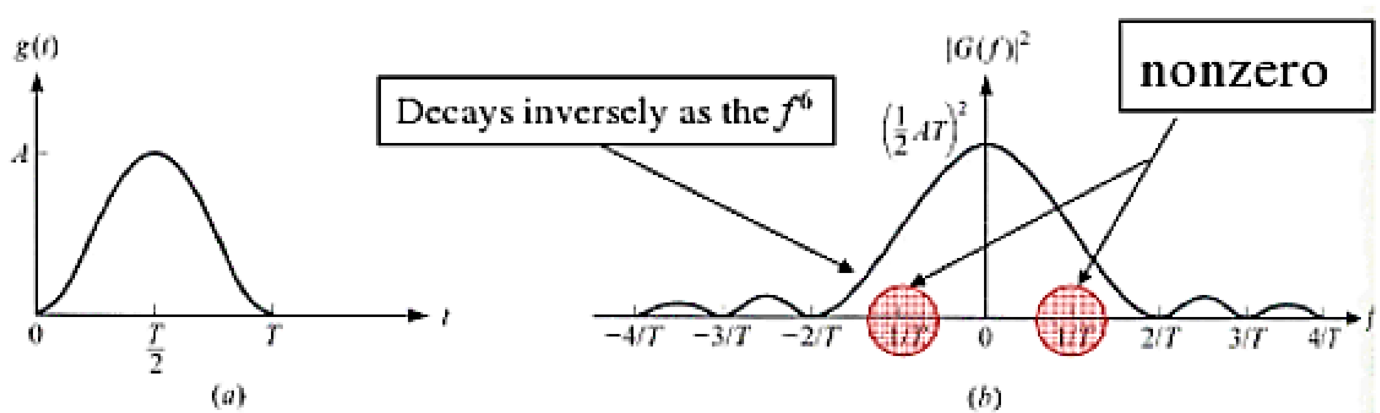
Example III

A second illustration of the spectral shapeing

- Raised cosine pulse

$$g(t) = \frac{A}{2}\left[1 + \cos\frac{2\pi}{T}(t - \frac{T}{2})\right] \qquad 0 \le t \le T$$

$$G(f) = \frac{AT}{2}\,\frac{\sin \pi f T}{\pi f\, T\,(1 - f^2 T^2)}\, e^{-j\pi f T}$$



- Has zeros at $f = \frac{n}{T}$, $n = \pm 2, \pm 3$, ... all discrete spectral components except the ones at $f = 0$ and $f = \pm\frac{1}{T}$ vanishes.
- Broader main lobe, but the tail decay inversely f^6
- Which ones uses less bandwidth?
  - It depends on the definition of the bandwidth.

Example 4: Controlling Spectrum by Operations on the info. Sequence

- $\{b_n\} \rightarrow$ binary sequence , b_n: uncorrelated $-1, +1$, $\mu = 0$, $\sigma^2 = 1$
- $I_n = b_n + b_{n-1}$

$$R_I = E[I_n I_{n+m}] = \begin{cases} 2 & m = 0 \\ 1 & m = \pm 1 \\ 0 & otherwise \end{cases}$$

How?

$$S_I(f) = 2(1 + \cos 2\pi f T) = 4\cos^2 \pi f T$$

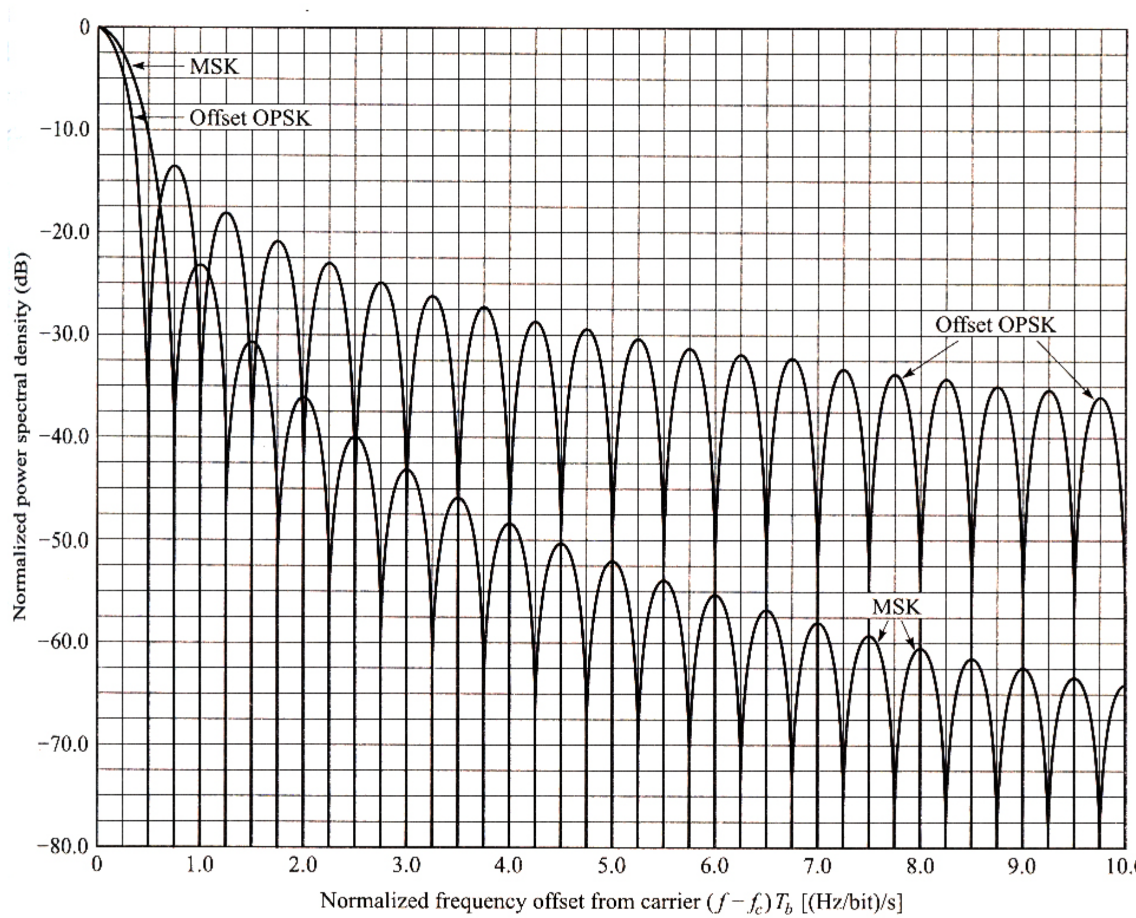
$$S_v(f) = \frac{4}{T} |G(f)|^2 \cos^2 \pi f T$$

Check the provided MATLAB code and the presentation of the results

Power Spectrum of CPFSK and CPM signals

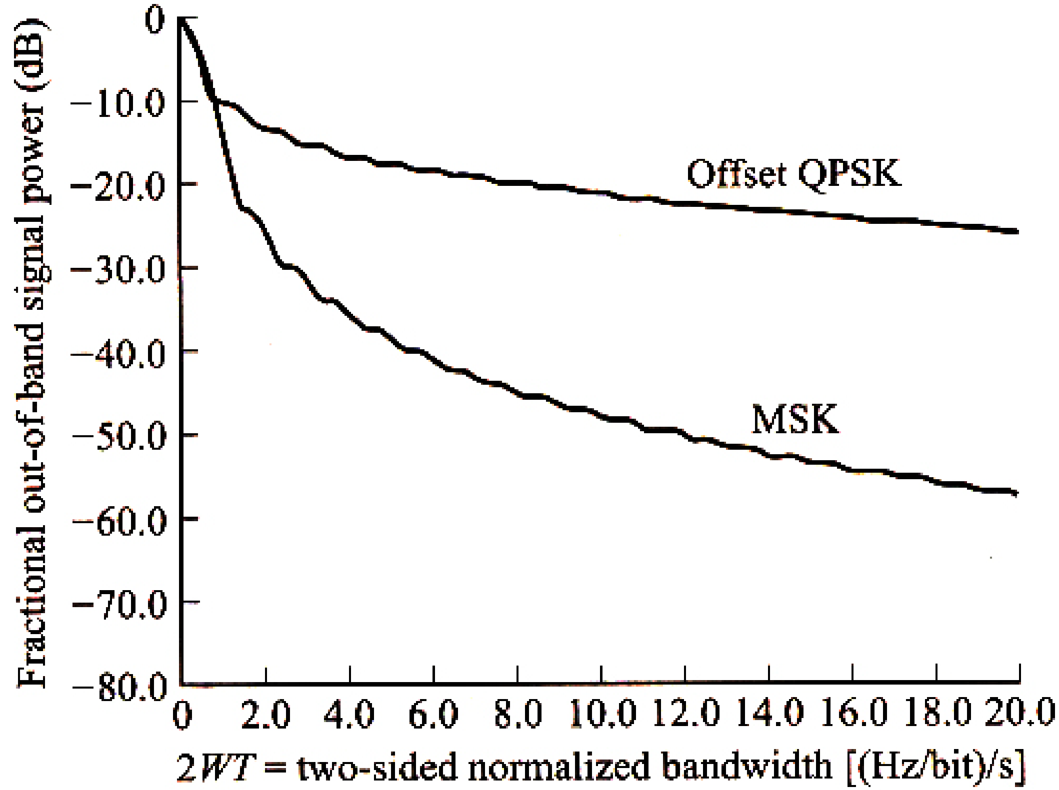
See figure 3.4.4

The spectra of the MSK and OQPSK the main lobe of MSK is 50% wider. However the side lobes of MSK fall off considerably faster.



Fractional out of band Power

MSK offers better fractional out-of-band power above fT_b=1. This is why it is popular in many communication systems



- The spectra of the MSK and OQPSK the main lobe of MSK is 50% wider. However the side lobes of MSK fall off considerably faster.
- 99% power
  - $W = 1.2/T_p$ for MSK
  - $W \approx 8/T_p$ for OQPSK

- FSK efficiency can be improved (but will lose orthogonality)
- There is special issue on bandwidth-efficient modulation and coding published by the IEEE Transaction on communication (March 1981) CPM?