Ch2: Probability & Stochastic Processes

- Importance of statistics to Communications
  Source info. is random ?! "to the system designer"
  noise, Interference ... random

- Previous exposure is expected... we do brief review
  "It is the responsability of the student to review.

2.1 Probability

- Example (Experiment, sample Spase S, outcomes, Event A, $P(A)$ Probability
  $\cup$ union, $\cap$ intersection,

Two events are **mutually exclusive** $\Rightarrow$ the intersection is the null event $\emptyset$

- Joint events & their probability. $P(A,B)$
- Conditional probability $P(A|B) = \dfrac{P(A,B)}{P(B)}$ Baye Rule

or $\boxed{P(A,B) = P(A|B)\,P(B) = P(B|A)\,P(A)}$

- Statistical ~~Indepet~~ independance $P(A|B) = P(A)$
  $P(A,B) = P(A)\,P(B)$
  $P(A,B,C) = P(A)\,P(B)\,P(C)$

2.1.1 Random Variable

- Represent the outcomes in terms of numbers $X(s)$

We define the CDF (Cumulative distribution function)

$F(x) = P(X \leq x) \qquad -\infty < x < \infty \qquad F(-\infty) = 0$
$F(\infty) = 1$

the PDF (probability density function)

$$p(x) = \frac{d\,F(x)}{dx} \qquad -\infty < x < \infty \qquad\qquad F(x) = \int_{-\infty}^{x} p(u)\,du \qquad -\infty < x < \infty$$

$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1) = \int_{x_1}^{x_2} p(x)\,dx$$

- For Random Variables, we may define
  joint & conditional pdf's & CDF's
- Similarly for statistical independance

$$F(x_1, x_2, \ldots x_n) = F(x_1)\,F(x_2) \ldots F(x_n)$$

$$p(x_1, x_2 \ldots x_n) = p(x_1)\,p(x_2) \ldots p(x_n)$$

2.1.2. Functions of Random Variables . e.g. $Y = aX + b$

Given the CDF of X can we find the CDF of Y ?

2.1.3 Statistical Averages

mean expected value $E(X) \equiv m_x = \int_{-\infty}^{\infty} x\,p(x)\,dx$ "first moment"

In general the n^{th} moment

$$E[X^n) = \int_{-\infty}^{\infty} x^n\,p(x)$$

n^{th} Central moment

$$E\left[(x - m_x)^n\right] = \int_{-\infty}^{\infty} (x - m_x)^n\,p(x)\,dx$$

Variance $\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2\,p(x)\,dx$

$$\sigma_x^2 = E(X^2) - [E(X)]^2 = E(X^2) - m_x^2$$

standard deviation $= \sigma_x$

2.1.4 Some Useful Probability Distribution

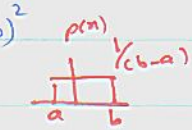
[1]. Binomial distribution $P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$

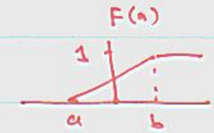
read text

[2] Uniform distribution between a & b

$$E(X) = \frac{1}{2}(a+b) \qquad \sigma^2 = \frac{1}{12}(a-b)^2$$

? sketch the pdf & CDF?





[3] Gaussian (normal) distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m_x)^2/2\sigma^2}$$

$$F(x) = \int_{-\infty}^{x} p(u)\,du = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-m_x}{\sqrt{2}\,\sigma}\right)$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$

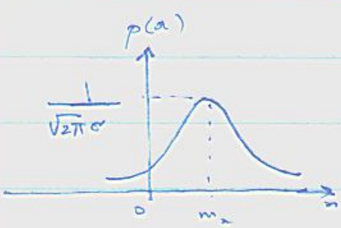
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

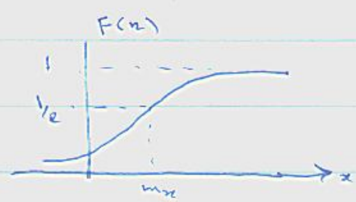
* notice erf, erfc

Q function.

different definitions.

$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$





other important distributions.

[4] Chi-square $Y = X_1^2 + X_2^2$

[5] Rayleigh distribution $Y = \sqrt{X_1^2 + X_2^2}$

where X_1 and X_2 are zero-mean statistically independent Gaussian R.V.

[6] Rice distribution

7 Nakagami m-distribution

8 Lognormal distribution

9 Multivariate Gaussian distribution see p. 48

2.1.5 Upper Bounds on the Tail Probability

It is often necessary to determine the area under the tail of the PDF

Chebyshev inequality $P(|X - m_x| \geq \delta) \leq \dfrac{\sigma_x^2}{\delta^2}$

for any positive number δ

Chernoff bound "one tail bound"

2.1.6 Sum of R.Vs and the Central Limit Theorem

very important

the sum of statistically independent & identically distributed r.v.s with finite mean and variance approaches a Gaussian CDF as $n \to \infty$

Can be illustrated using Matlab (R)

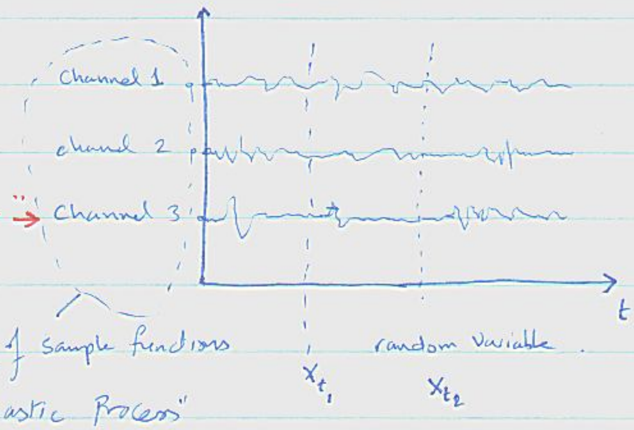
2.2. STOCHASTIC PROCESSES

p.61 →

. Define →

an ensample of sample functions

"Sample function." →



. Many random phenomina are functions of time. "importance" → noise, source, channel.

. r.v.s are charachterized by pdf

. r. processes are " " joint pdf $p(x_{t_1}, x_{t_2}, \cdots)$

Def. Stationary stochastic Process

$$p(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) = p(X_{t_1+t}, x_{t_2+t}, \cdots x_{t_n+t})$$

for all t and all n "stationary in the strict sense"

other-wise "nonstationary"

2.2.1 Statistical Averages.

ensamble averages

generally depend on t_i

n th moment $\quad E(X_{t_i}^n) = \int_{-\infty}^{\infty} x_{t_i}^n p(x_{t_i})\, dx_{t_i}$

Auto correlation $\phi(t_1, t_2)$

joint moment $\quad = E(X_{t_1}, X_{t_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_{t_1} x_{t_2}\, p(X_{t_1}, X_{t_2})\, dX_{t_1}\, dX_{t_2}$

For Stationary processes

$$E(X_{t_1}, X_{t_2}) = \phi(t_1, t_2) = \phi(t_1 - t_2) = \phi(\tau)$$

$$\phi(\tau) = \phi(-\tau) \qquad \text{even function}$$

$$\phi(0) = E(X_t^2) \equiv \text{average power}$$

Def: . The nonstationary process with

1) Constant mean

2) $\phi(t_1, t_2) = \phi(t_1 - t_2)$

is called $\boxed{\text{Wide-sense stationary}}$ WSS

. Usually if we say stationary without specification we refer to WSS

Auto covariance

$$\mu(t_1, t_2) = E\{[X_{t_1} - m(t_1)][X_{t_2} - m(t_2)]\}$$

$$= \phi(t_1, t_2) - m_1(t_1) m(t_2)$$

for stationary process

$$\mu(t_1, t_2) = \mu(t_1 - t_2) = \mu(\tau) = \phi(\tau) - m^2$$

A Gaussian Random Process is completly characterized by its first two moments.

For Gaussian r.p. WSS $\Rightarrow$ strict sense stationarity

Averages for joint stochastic Processes

Let $X(t)$, $Y(t)$ be two random processes defined by the joint pdf $p(x_{t_1}, x_{t_2}, \ldots x_{t_n}, y_{t_1'}, y_{t_2'}, \ldots y_{t_n'})$

Cross-correlation function

$$\phi_{XY}(t_1, t_2) = E[X_{t_1} Y_{t_2}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_{t_1} Y_{t_2}\, P(x_{t_1}, y_{t_2})\, dX_{t_1}\, dX_{t_2}$$

Cross-covariance $\mu_{XY}(t_1, t_2) = \phi_{XY}(t_1, t_2) - m_x(t_1) m_y(t_2)$

$X(t)$ & $Y(t)$ are <u>statistically independent</u>

$$p(x_{t_1}, x_{t_2}, \ldots x_{t_n}, y_{t_1'}, y_{t_2'}, \ldots) = p(x_{t_1}, x_{t_2} \ldots)\, p(y_{t_1'}, y_{t_2'} \ldots y_{t_n'})$$

Uncorrelated if

$$\phi_{xy}(t_1, t_2) = E(X_{t_1}) E(Y_{t_2}) \Rightarrow \mu_{xy}(t_1, t_2) = 0$$

Complex-valued stochastic process $Z(t)$

$$Z(t) = X(t) + j\,Y(t)$$

arbitrary

$$\phi_{zz}(t_1, t_2) = \frac{1}{2} E(Z_{t_1} Z^*_{t_2}) \qquad \text{auto correlation}$$

$$= \frac{1}{2} E\left[(X_{t_1} + j Y_{t_1})(X_{t_2} - j Y_{t_2})\right]$$

$$= \frac{1}{2} \left\{ \phi_{xx}(t_1, t_2) + \phi_{yy}(t_1, t_2) + j\left[\phi_{yx}(t_1, t_2) - \phi_{xy}(t_1, t_2)\right]\right\}$$

2.2.2 Power Density Spectrum (Power Spectral Density PSD)

. A stationary stochastic process is an infinite energy signal
(No Fourier Transform) , PSD $\Phi(f)$

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \mathcal{F}[\phi(\tau)]$$

$$\phi(\tau) = \int_{-\infty}^{\infty} \Phi(f)\, e^{j2\pi f\tau}\, df$$

Note $\phi(0) = \int_{-\infty}^{\infty} \Phi(f)\, df = E(|X_t|^2) \geq 0$ average power

area under

Φ_{xy} cross-power density spectrum. $\Phi(f)$ or $\phi(\tau)$

2.2.3 Response of a linear Time-Invariant System to a Random input Signal.

$$y = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$m_y = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)]\, d\tau$$

$$= m_x \int_{-\infty}^{\infty} h(\tau)\, d\tau = m_x H(0) \qquad \boxed{m_y = m_x H(0)} \text{ — (1)}$$

$$\phi_{yy}(t_1, t_2) = \frac{1}{2} E\left(Y_{t_1} Y^*_{t_2}\right)$$ auto correlation.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\beta)\, h^*(\alpha)\, \phi_{xx}(t_1 - t_2 + \alpha - \beta)\, d\alpha\, d\beta$$

$$\boxed{\Phi_{yy}(f) = \Phi_{xx}(f)\, |H(f)|^2}$$

power density spectrum.

$$\phi_{yy}(\tau) = \int_{-\infty}^{\infty} \Phi_{yy}(f)\, e^{j2\pi f\tau}\, df$$
$$= \int_{-\infty}^{\infty} \Phi_{xx}(f)\, |H(f)|^2\, e^{j2\pi f t}\, df$$

Example 2.2.-1 p 68

⇒ The Concepts are Extended to Discrete-Time process $X(n)$ [2.2.5 p71]

For Example $\Phi(f) = \sum_{n=-\infty}^{\infty} \phi(n)\, e^{-j2\pi f n}$

expl p7

2.2.6 Cyclostationary Process

mean & autocorrelation are periodic with period $= T$

$$X(t) = \sum_{n=-\infty}^{\infty} a_n\, g(t - nT)$$

a_n: random

widly used to represent sequences of data with rate $= \frac{1}{T}$

White Noise : "Examples for R.P. Through linear Sys"

- Flat over all frequencies "IDEAL"

$$\Phi_w(f) = \frac{N_0}{2} \quad \text{Watts/Hz}$$

$$\phi_w(\tau) = \frac{N_0}{2}\,\delta(\tau) \qquad \text{By inverse Fourier Transform}$$

- Any two samples, no matter how closely together in time they are taken, are uncorrelated.
- if they are Gaussian ⇒ independent "statistically"

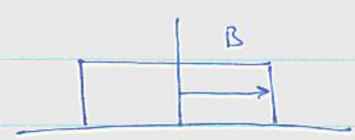
WGN represents the highest degree in "randomness"

- White noise as average power $\phi_w(0) = \frac{N_0}{2}\,\delta(0)$
  - → mathematically easy to deal with
  - → Can be approximated for a given bandwidth

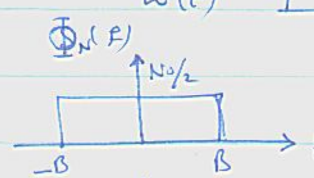
Example I :

Ideal low Pass filtered noise :

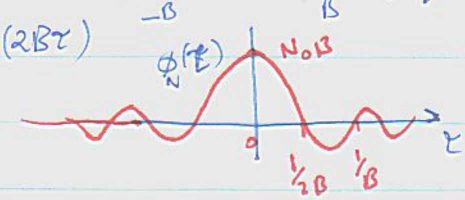
Bandwidth = B amplitude = 1



$$\Phi_N(f) = \begin{cases} \frac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases}$$



$$\phi_N(\tau) = \int_{-B}^{B} \Phi_N(f)\, e^{j2\pi f\tau}\, df = N_0 B\, \text{sinc}(2B\tau)$$



The output is also Gaussian. Gaussian $\rightarrow$ Linear filter $\rightarrow$ Gaussian

are correlated except samples taken at $\tau = \pm \frac{k}{2B}$

are uncorrelated "Because they are Gaussian $\Rightarrow$ statistically independent"

$$\mu_N = \mu_W \, H(0) = 0 \qquad (\mu_W = 0)$$

Watt/Hz . Hz

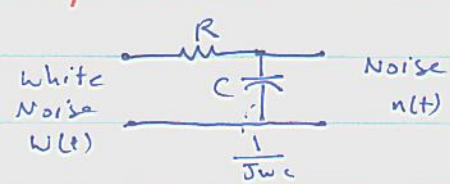
$$\sigma_N^2 = E\left[n^2(t)\right] - \mu_N^2 = \phi_N(0) = N_0 B \text{ watts}$$

Example II

RC low-pass filtered white Noise

A white Gaussian Noise $W(t)$ with zero mean and PSD $\frac{N_0}{2}$ is applied to a low-pass RC filter.

$$H(f) = \frac{1}{1 + j2\pi f RC}$$



we get $H(f)$ using voltage divider. $\frac{1/j\omega c}{R + \frac{1}{j\omega c}}$

$$\Phi_N(f) = \Phi_W(f) \, |H(f)|^2 = \frac{N_0/2}{1 + (2\pi f RC)^2}$$

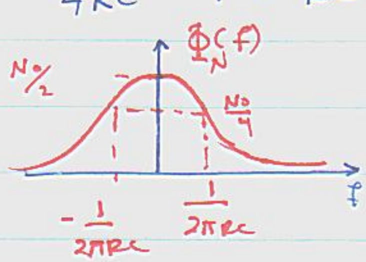
By inverse Fourier Transform "Tables"

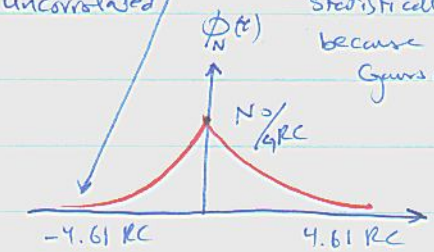
$$\phi_N(\tau) = \frac{N_0}{4RC} \exp\left(-\frac{|\tau|}{RC}\right)$$

if the noise at the output is sampled at a rate $\leq \frac{1}{4.61\,RC}$ samples/sec

the resulting samples are essentially uncorrelated

"statistically indep. because of being Gaussian"





A Natural Example that follows next is

Narrow-band noise:

This requires some mathematical representation which is the motive for Chapter 4 "Signal Representation"