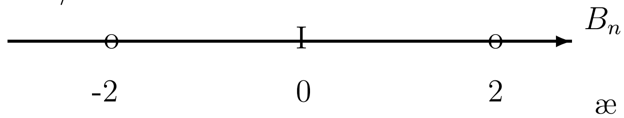
Problem 4.21 :

(a) $B_n = I_n + I_{n-1}$. Hence :

I_n	I_{n-1}	B_n
1	1	2
1	-1	0
-1	1	0
-1	-1	-2

The signal space representation is given in the following figure, with $P(B_n = 2) = P(B_n = -2) = 1/4, \quad P(B_n = 0) = 1/2$.



(b)

$$\begin{aligned} \phi_{BB}(m) &= E\left[B_{n+m}B_n\right] = E\left[\left(I_{n+m} + I_{n+m-1}\right)\left(I_n + I_{n-1}\right)\right] \\ &= \phi_{ii}(m) + \phi_{ii}(m-1) + \phi_{ii}(m+1) \end{aligned}$$

Since the sequence $\{I_n\}$ consists of independent symbols :

$$\phi_{ii}(m) = \left\{ \begin{array}{cc} E\left[I_{n+m}\right] E\left[I_n\right] = 0 \cdot 0 = 0, & \text{m} \neq 0 \\ E\left[I_n^2\right] = 1, & \text{m} = 0 \end{array} \right\}$$

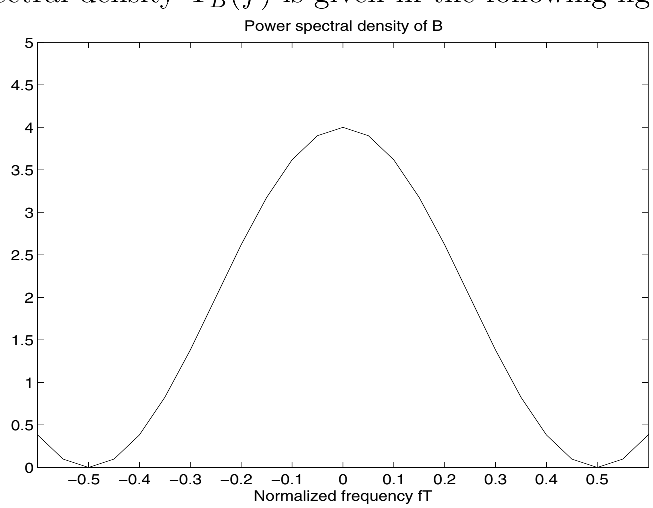
Hence :

$$\phi_{BB}(m) = \left\{ \begin{array}{cc} 2, & \text{m} = 0 \\ 1, & \text{m} = \pm 1 \\ 0, & \text{o.w} \end{array} \right\}$$

and

$$\begin{aligned} \Phi_{BB}(f) &= \sum_{m=-\infty}^{\infty} \phi_{BB}(m) \exp(-j2\pi f m T) = 2 + \exp(j2\pi f T) + \exp(-j2\pi f T) \\ &= 2\left[1 + \cos 2\pi f T\right] = 4\cos^2 \pi f T \end{aligned}$$

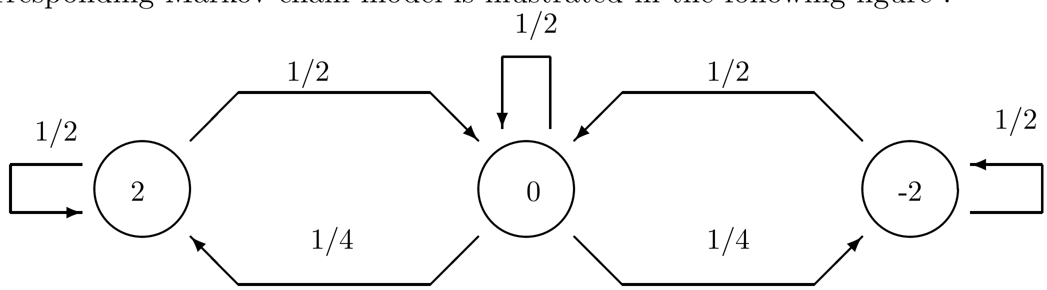
A plot of the power spectral density $\Phi_B(f)$ is given in the following figure :



(c) The transition matrix is :

I_{n-1}	I_n	B_n	I_{n+1}	B_{n+1}
-1	-1	-2	-1	-2
-1	-1	-2	1	0
-1	1	0	-1	0
-1	1	0	1	2
1	-1	0	-1	-2
1	-1	0	1	0
1	1	2	-1	0
1	1	2	1	2

The corresponding Markov chain model is illustrated in the following figure :



æ

Problem 4.22 :

(a) $I_n = a_n - a_{n-2}$, with the sequence $\{a_n\}$ being uncorrelated random variables (i.e $E\left(a_{n+m}a_n\right) = \delta(m)$). Hence :

$$\begin{aligned}\phi_{ii}(m) &= E\left[I_{n+m}I_n\right] = E\left[\left(a_{n+m} - a_{n+m-2}\right)\left(a_n - a_{n-2}\right)\right] \\ &= 2\delta(m) - \delta(m-2) - \delta(m+2) \\ &= \left\{\begin{array}{cc} 2, & \text{m} = 0 \\ -1, & \text{m} = \pm 2 \\ 0, & \text{o.w.}\end{array}\right\}\end{aligned}$$

(b) $\Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2 \Phi_{ii}(f)$ where :

$$\begin{aligned}\Phi_{ii}(f) &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m)\exp(-j2\pi f mT) = 2 - \exp(j4\pi fT) - \exp(-j4\pi fT) \\ &= 2\left[1 - \cos 4\pi fT\right] = 4\sin^2 2\pi fT\end{aligned}$$

and

$$\left|G(f)\right|^2 = (AT)^2\left(\frac{\sin \pi fT}{\pi fT}\right)^2$$

Therefore :

$$\Phi_{uu}(f) = 4A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin^2 2\pi fT$$

(c) If $\{a_n\}$ takes the values (0,1) with equal probability then $E(a_n) = 1/2$ and $E(a_{n+m}a_n) = \left\{ \begin{array}{ll} 1/4, & \mathrm{m} \neq 0 \\ 1/2, & \mathrm{m} = 0 \end{array} \right\} = [1 + \delta(m)]/4$. Then :

$$\begin{array}{rcl} \phi_{ii}(m) & = & E\left[I_{n+m}I_n\right] = 2\phi_{aa}(0) - \phi_{aa}(2) - \phi_{aa}(-2) \\ & = & \frac{1}{4}\left[2\delta(m) - \delta(m-2) - \delta(m+2)\right] \end{array}$$

and

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \exp(-j2\pi fmT) = \sin^2 2\pi fT$$
$$\Phi_{uu}(f) = A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin^2 2\pi fT$$

Thus, we obtain the same result as in (b) , but the magnitude of the various quantities is reduced by a factor of 4 .

Problem 4.35 :

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}, \quad \rho = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

By straightforward matrix multiplication we verify that indeed :

$$\mathbf{P}^4 \rho = -\frac{1}{4}\rho$$

CHAPTER 5

Problem 5.1 :

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \mathrm{sgn}(t) - \mathrm{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

where $sgn(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

(b) The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 5.5 :

Since $\{f_n(t)\}$ constitute an orthonormal basis for the signal space : $r(t) = \sum_{n=1}^{N} r_n f_n(t), \;\; s_m(t) =$

$\sum_{n=1}^{N} s_{mn} f_n(t)$. Hence, for any m :

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= 2\int_0^T r(t) s_m(t) dt - \int_0^T s_m^2(t) dt \\ &= 2\int_0^T \sum_{n=1}^{N} r_n f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt - \int_0^T \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt \\ &= 2\sum_{n=1}^{N} r_n \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt - \sum_{n=1}^{N} s_{mn} \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt \\ &= 2\sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2 \end{aligned}$$

where we have exploited the orthonormality of $\{f_n(t)\}$: $\int_0^T f_n(t) f_l(t) dt = \delta_{nl}$. The last form is indeed the original form of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$.

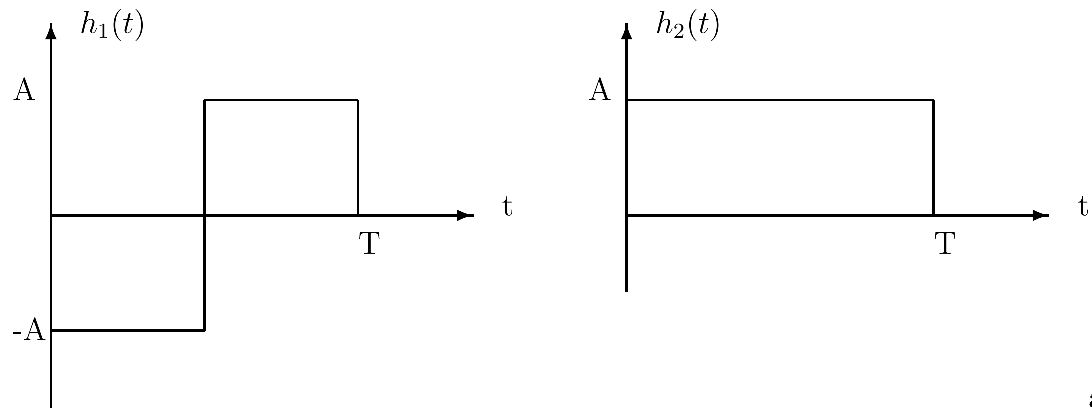
Problem 5.8 :

(a) Since the given waveforms are the equivalent lowpass signals :

$$\mathcal{E}_1 = \frac{1}{2}\int_0^T |s_1(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$
$$\mathcal{E}_2 = \frac{1}{2}\int_0^T |s_2(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$

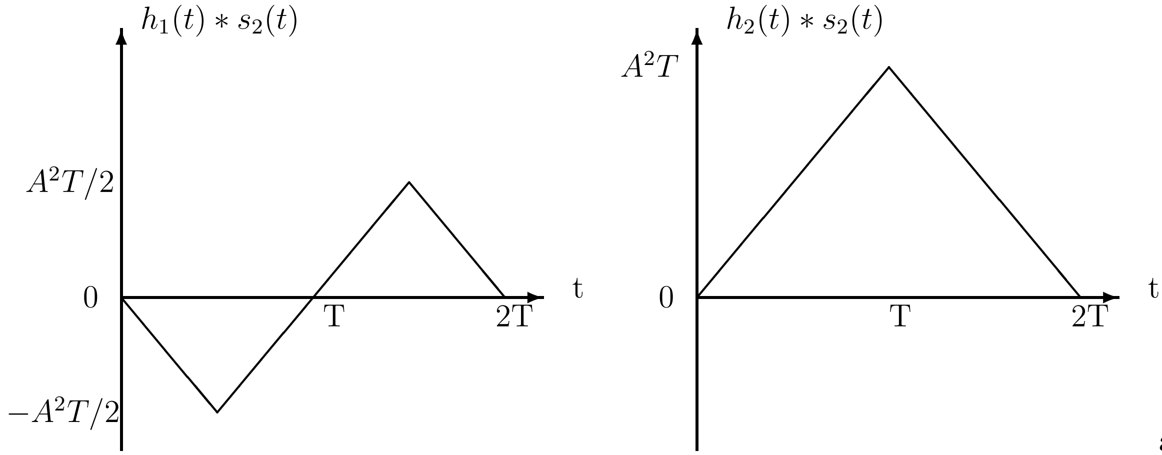
Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}} \int_0^T s_1(t) s_2^*(t) dt = 0$.

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T-t)$. The following figure shows $h_i(t)$:



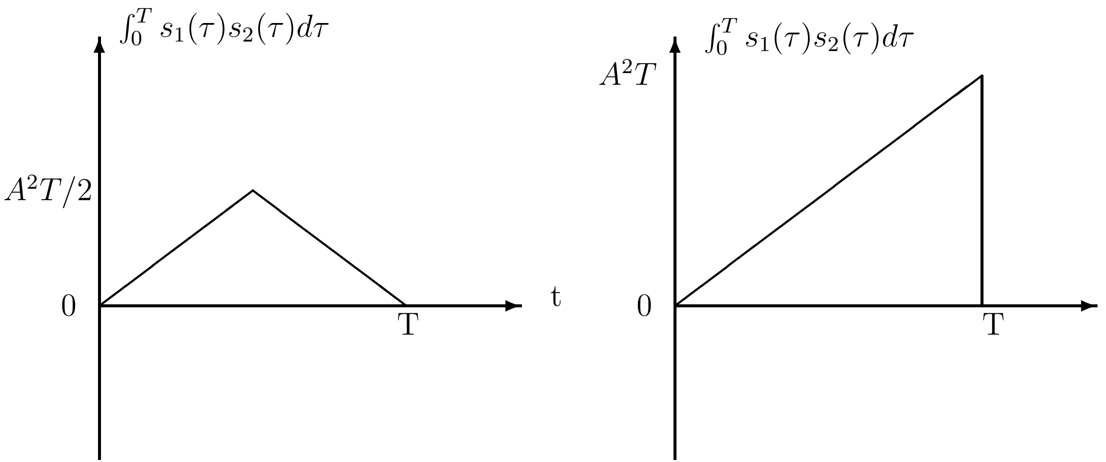
æ

(c)



æ

(d)



æ

(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.