otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\mathrm{PM}(\mathbf{r},\mathbf{s}_0)}{\mathrm{PM}(\mathbf{r},\mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2-r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \begin{array}{c} s_0 \\ > \\ < \\ s_1 \end{array} 1$$

or equivalently :

$$r \begin{array}{c} s_1 \\ > \\ < \\ s_0 \end{array} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\ &= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\ &= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\ &= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right] \end{aligned}$$

where

$$\mathrm{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 5.5 :

Since $\{f_n(t)\}$ constitute an orthonormal basis for the signal space : $r(t) = \sum_{n=1}^{N} r_n f_n(t),\ s_m(t) =$

$\sum_{n=1}^{N} s_{mn} f_n(t)$. Hence, for any m :

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= 2\int_0^T r(t)s_m(t)dt - \int_0^T s_m^2(t)dt \\ &= 2\int_0^T \sum_{n=1}^{N} r_n f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t)dt - \int_0^T \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t)dt \\ &= 2\sum_{n=1}^{N} r_n \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t)f_l(t)dt - \sum_{n=1}^{N} s_{mn} \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t)f_l(t)dt \\ &= 2\sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2 \end{aligned}$$

where we have exploited the orthonormality of $\{f_n(t)\}$: $\int_0^T f_n(t)f_l(t)dt = \delta_{nl}$. The last form is indeed the original form of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$.

Problem 5.6 :

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E\left[|n(T)|^2\right]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{aligned} E\left[|n(T)|^2\right] &= \int_0^T \int_0^T E\left[z(a)z^*(b)\right] h_l(T-a)h_l^*(T-b)dadb \\ &= 2N_0 \int_0^T |h_l(T-t)|^2 dt \end{aligned}$$

so we want to maximize :

$$SNR = \frac{\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2}{2N_0 \int_0^T |h_l(T-t)|^2 dt}$$

From Schwartz inequality :

$$\left|\int_0^T s_l(t)h_l(T-t)dt\right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt$$

Hence :

$$SNR \leq \frac{1}{2N_0}\int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$

Problem 5.7 :

$$N_{mr} = Re\left[\int_0^T z(t) f_m^*(t)dt\right]$$

(a) Define $a_m = \int_0^T z(t) f_m^*(t)dt$. Then, $N_{mr} = Re(a_m) = \frac{1}{2}\left[a_m + a_m^*\right].$

$$E(N_{mr}) = Re\left[\int_0^T E\left(z(t)\right) f_m^*(t)dt\right] = 0$$

since, $E\left[z(t)\right] = 0$. Also :

$$E\left(N_{mr}^2\right) = E\left[\frac{a_m^2 + (a_m^*)^2 + 2a_m a_m^*}{4}\right]$$

But $E\left(a_m^2\right) = E\left[\int_0^T\int_0^T z(a)z(b) f_m^*(a) f_m^*(b) da db\right] = 0$, since $E\left[z(a)z(b)\right] = 0$ (Problem 4.3), and the same is true for $E\left[(a_m^*)^2\right] = 0$, since $E\left[z^*(a)z^*(b)\right] = 0$ Hence :

$$\begin{array}{rcl} E\left(N_{mr}^2\right) & = & E\left[\frac{a_m a_m^*}{2}\right] = \frac{1}{2}\int_0^T\int_0^T E\left[z(a)z^*(b)\right] f_m^*(a) f_m(b) da db \\ & = & N_0 \int_0^T \left|f_m(a)\right|^2 da = 2\mathcal{E} N_0 \end{array}$$

(b) For $m \neq k$:

$$\begin{array}{rcl} E\left[N_{mr} N_{kr}\right] & = & E\left[\frac{a_m + a_m^*}{2}\frac{a_k + a_k^*}{2}\right] \\ & = & E\left[\frac{a_m a_k + a_m^* a_k + a_m a_k^* + a_m^* a_k^*}{4}\right] \end{array}$$

But, similarly to part (a), $E\left[a_m a_k\right] = E\left[a_m^* a_k^*\right] = 0$, hence, $E\left[N_{mr} N_{kr}\right] = E\left[\frac{a_m^* a_k + a_m a_k^*}{4}\right]$. Now :

$$\begin{array}{rcl} E\left[a_m a_k^*\right] & = & \int_0^T\int_0^T E\left[z(a)z^*(b)\right] f_m^*(a) f_k(b) da db \\ & = & 2N_0 \int_0^T f_m^*(a) f_k(a) da = 0 \end{array}$$

since, for $m \neq k$, the waveforms are orthogonal.
Similarly : $E\left[a_m^* a_k\right] = 0$, hence : $E\left[N_{mr} N_{kr}\right] = 0$.

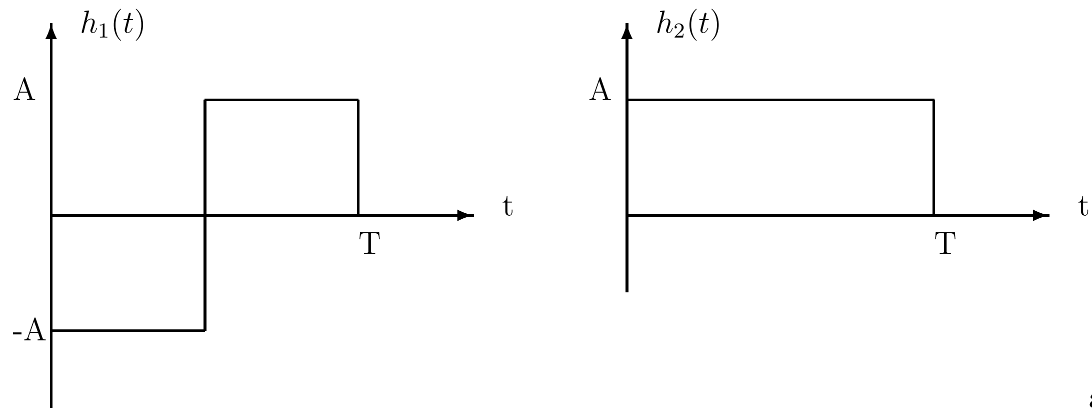
Problem 5.8 :

(a) Since the given waveforms are the equivalent lowpass signals :

$$\begin{array}{l} \mathcal{E}_1 = \frac{1}{2}\int_0^T \left|s_1(t)\right|^2 dt = \frac{1}{2}A^2 \int_0^T dt = A^2 T/2 \\ \mathcal{E}_2 = \frac{1}{2}\int_0^T \left|s_2(t)\right|^2 dt = \frac{1}{2}A^2 \int_0^T dt = A^2 T/2 \end{array}$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}} \int_0^T s_1(t) s_2^*(t) dt = 0$.

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T-t)$. The following figure shows $h_i(t)$:



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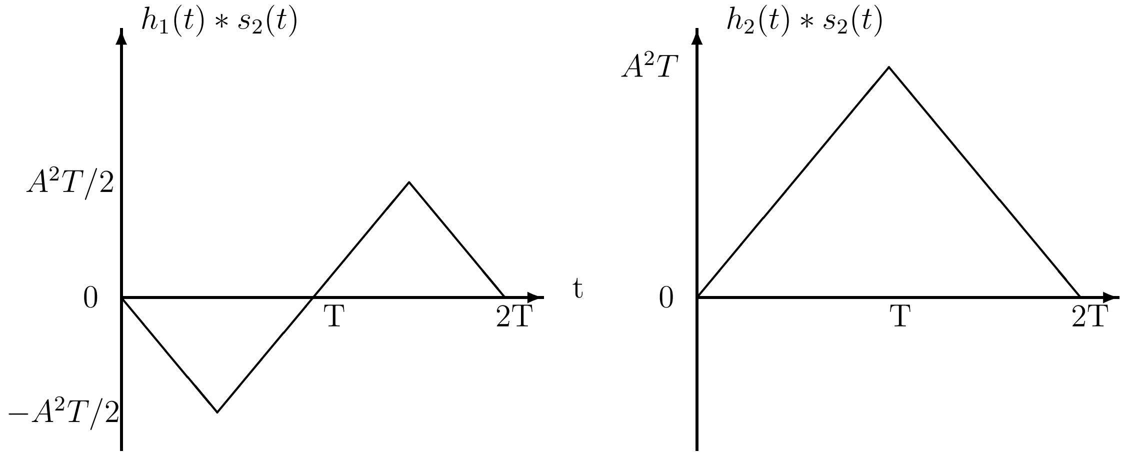
(c)



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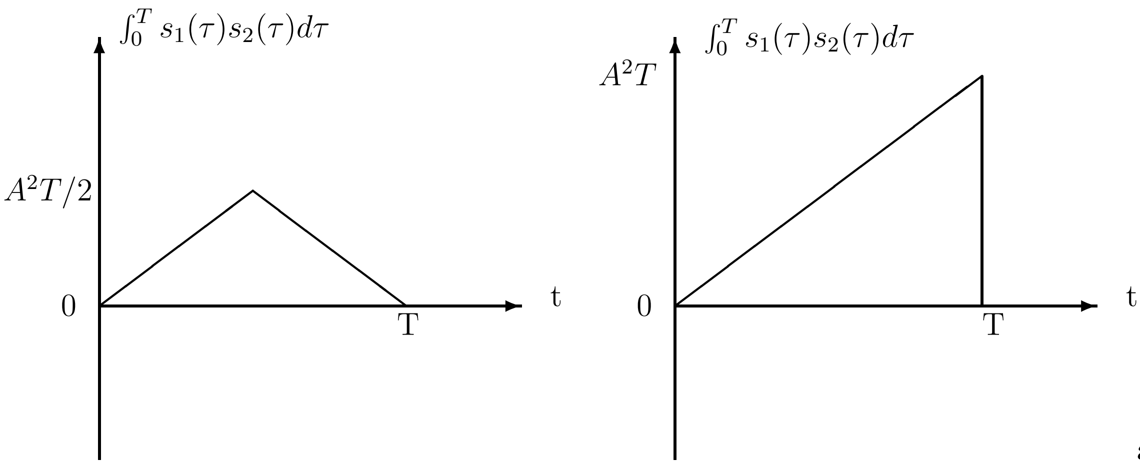
(d)



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(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

Problem 5.9 :

(a) The joint pdf of a, b is :

$$p_{ab}(a,b) = p_{xy}(a - m_r, b - m_i) = p_x(a - m_r)p_y(b - m_i) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(a-m_r)^2+(b-m_i)^2\right]}$$

(b) $u = \sqrt{a^2 + b^2}$, $\quad \phi = \tan^{-1} b/a \Rightarrow a = u\cos\phi,\ b = u\sin\phi$ The Jacobian of the transformation is : $J(a,b) = \begin{vmatrix} \partial a/\partial u & \partial a/\partial\phi \\ \partial b/\partial u & \partial b/\partial\phi \end{vmatrix} = u$, hence :

$$\begin{aligned} p_{u\phi}(u,\phi) &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(u\cos\phi - m_r)^2+(u\sin\phi - m_i)^2\right]} \\ &= \frac{u}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[u^2+M^2-2uM\cos(\phi-\theta)\right]} \end{aligned}$$

where we have used the transformation :

$$\left\{ \begin{array}{l} M = \sqrt{m_r^2 + m_i^2} \\ \theta = \tan^{-1} m_i/m_r \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} m_r = M\cos\theta \\ m_i = M\sin\theta \end{array} \right\}$$