Wideband FM/PM Signals

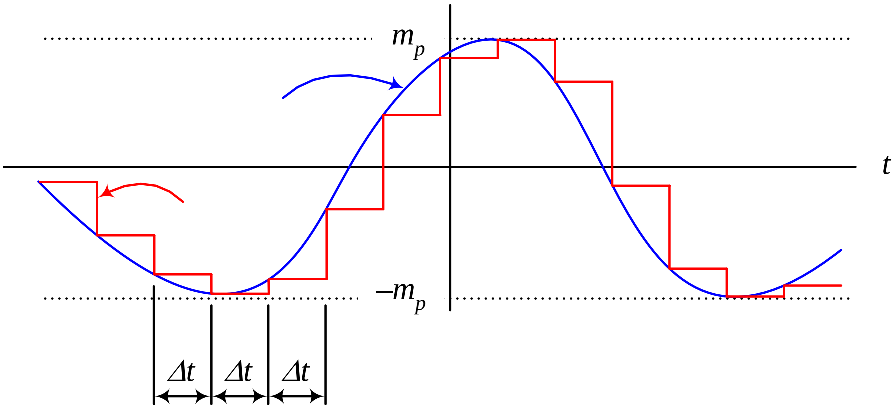
Contents

Introduction (Definition of Wideband FM/PM)

For the FM signal shown below, the value of k_f may not satisfy the condition $|k_f a(t)| \ll 1$, and therefore the approximation used in narrowband FM may not be applicable. For FM signals

$$g_{FM}(t) = A \cdot \cos\left[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha) d\alpha\right].$$

that do not satisfy $|k_f a(t)| \ll 1$, computing the bandwidth is more difficult than for narrowband signals. To compute the bandwidth of wideband FM, we will approximate the signal $g_{FM}(t)$ by another signal that results from modulating a sampled version of the message signal. That is, instead of using the signal $m(t)$ as the message signal for modulating the carrier, we will use an approximation $\hat{m}(t)$ that is obtained by sampling $m(t)$ as shown in the following figure.



The value of Δt that we will use is the maximum allowable value that will insure that the $m(t)$ can be reconstructed from $\hat{m}(t)$ without loss of information. This maximum value for Δt is obtained using the Nyquist sampling theorem, which states that for a signal $m(t)$ with a bandwidth of B_m (Hz), the minimum sampling frequency is $2B_m$. Therefore, the maximum for Δt is given by $\Delta t = 1/2B_m$. Of course a smaller value for Δt will be better since it gives a better approximation for $m(t)$ but is unnecessary.

So, to find the approximate bandwidth of FM signal, let us assume that the original message signal $m(t)$ is bounded in amplitude by the two values m_p and $-m_p$. Therefore,

$$-m_p \le m(t) \le m_p .$$

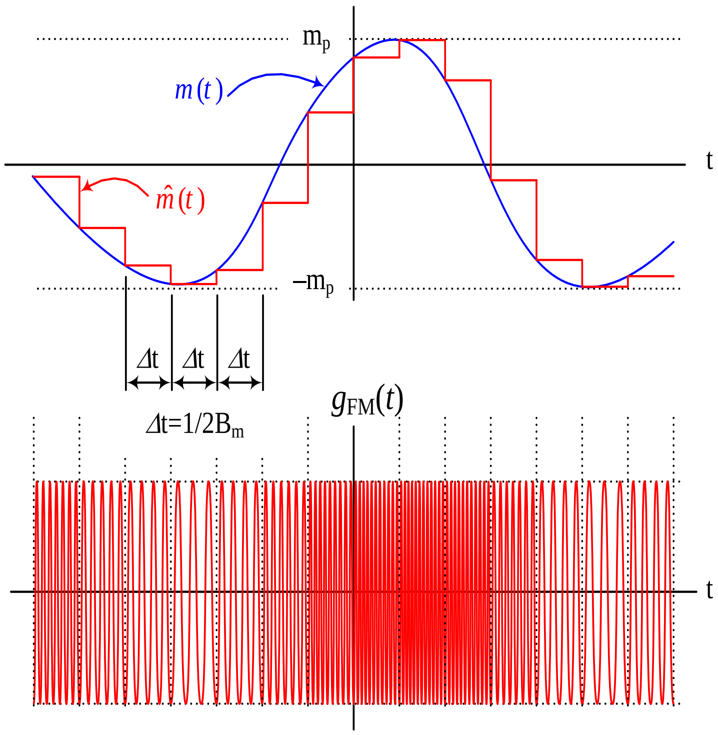
Then, we can straight forward say that

$$-m_p \le \hat{m}(t) \le m_p .$$

Now, the approximate signal $\hat{g}_{FM}(t)$ to $g_{FM}(t)$ is obtained by FM modulating $\hat{m}(t)$ as

$$\hat{g}_{FM}(t) = A \cdot \cos\left[\omega_c t + k_f \int_{-\infty}^{t} \hat{m}(\alpha) d\alpha\right]$$

Since $\hat{m}(t)$ is constant over periods of $\Delta t = 1/2B_m$, the instantaneous frequency of $\hat{g}_{FM}(t)$ will be constant over periods of $\Delta t = 1/2B_m$. The signal $\hat{g}_{FM}(t)$ will look like as follows.



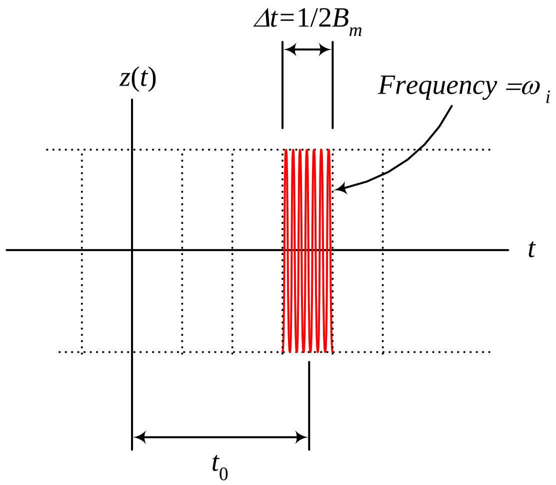
Since $-m_p \leq \hat{m}(t) \leq m_p$, the instantaneous frequency of $\hat{g}_{FM}(t)$ (and also $g_{FM}(t)$) will be in the range

$$\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p .$$

This means that the instantaneous frequency changes over a range of $\Delta\omega = k_f \cdot m_p$ (this can also be written as $\Delta f = k_f \cdot m_p / 2\pi$) on each side around the carrier frequency ω_c.

So, to approximate the bandwidth of the of original FM signal $g_{FM}(t)$, we will compute the approximate bandwidth of the approximation signal $\hat{g}_{FM}(t)$ by finding its frequency spectrum. Since $\hat{g}_{FM}(t)$ is composed of blocks of sinusoids with different frequencies that are in the range of frequencies of $\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p$, we can find the spectrum of each of these blocks independently and then add these spectrums to get the overall spectrum of $\hat{g}_{FM}(t)$.

Consider the part of $\hat{g}_{FM}(t)$ that shown below



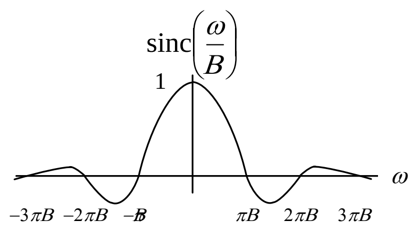
The signal $z(t)$ is given by

$$z(t) = A \cdot rect\left[2B_m(t - t_0)\right] \cdot \cos(\omega_i t),$$

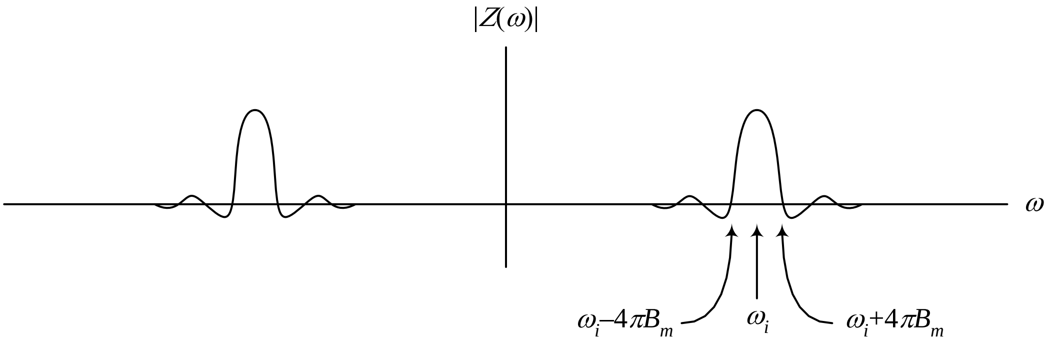
where $\omega_c - k_f m_p \leq \omega_i(t) \leq \omega_c + k_f m_p$. The Fourier transform of $z(t)$ is

$$Z(\omega) = \frac{A}{4B_m}\left[\text{sinc}\left(\frac{\omega - \omega_i}{4B_m}\right) \cdot e^{-j(\omega - \omega_i)t_0} + \text{sinc}\left(\frac{\omega + \omega_i}{4B_m}\right) \cdot e^{-j(\omega + \omega_i)t_0}\right].$$

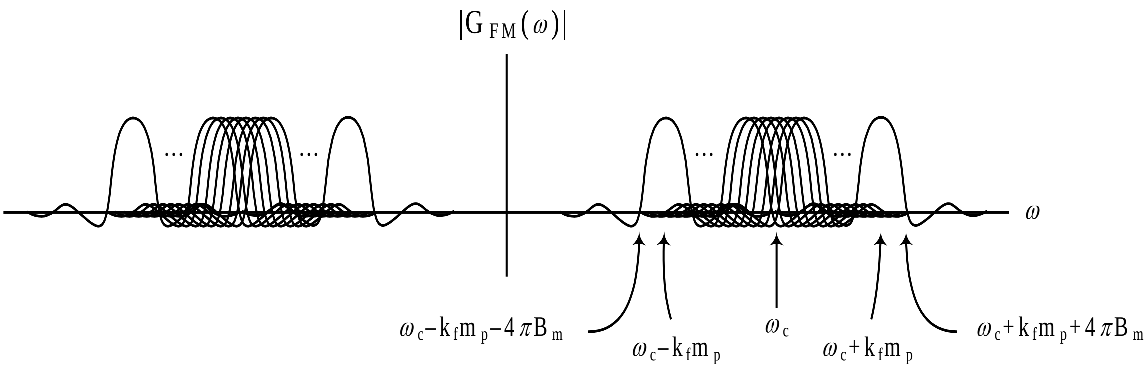
Remembering that the *sinc* function looks like the following



Sketching the magnitude of $Z(\omega)$ (i.e., magnitude spectrum of $z(t)$) will give the following (since the complex exponentials have a magnitude of one and the two components shifted to the left and right almost do not interfere with each other)



Adding the spectrum of the different signals like $z(t)$ given above will give us the spectrum of the approximation signal $\hat{g}_{FM}(t)$



If we assume that the sidebands (the small humps at the two edges of a *sinc* function) have negligible power, and knowing that $\Delta\omega = k_f \cdot m_p$, we see that the bandwidth of an FM signal is approximately equal to

$$\begin{aligned} BW_{FM} &\approx 2k_f \cdot m_p + 8\pi B_m \quad \text{(rad/s)} \\ &\approx 2 \cdot \Delta\omega + 8\pi B_m \quad \text{(rad/s)} \end{aligned}.$$

Using the fact that $\Delta f = k_f \cdot m_p / 2\pi$, the bandwidth in Hz becomes

$$\begin{aligned} BW_{FM} &\approx \frac{2k_f \cdot m_p}{2\pi} + 4B_m \quad \text{(Hz)} \\ &\approx 2 \cdot \Delta f + 4B_m \quad \text{(Hz)} \\ &\approx 2\left[\Delta f + 2B_m\right] \quad \text{(Hz)} \end{aligned}$$

In practice, this bandwidth is higher than the actual bandwidth of FM signals. Consider for example narrowband FM. Using this formula for the bandwidth, we see that the bandwidth is twice the actual bandwidth.

Carson's Rule

In fact, a more accurate relationship is known as CARSON's Rule, which is given by

$$\boxed{\begin{aligned} BW_{FM} &\approx 2 \cdot \Delta f + 2B_m \quad \text{(Hz)} \approx 2\left[\Delta f + B_m\right] \quad \text{(Hz)} \\ &\approx 2\left[\Delta\omega + 2\pi B_m\right] \quad \text{(rad/s)} \end{aligned}},$$

where

B_m = Bandwidth of the Message Signal $m(t)$ in Hz,

And $\Delta\omega = k_f \cdot m_p \rightarrow \Delta f = k_f \cdot m_p / 2\pi$.

Special Case

For very wideband FM $\Delta f \gg B_m$, Carson's rule can be approximated as

$BW_{FM} \approx 2 \cdot \Delta f \quad \text{(Hz)}$, this is the scenario that is compatible with the pioneers thinking)

The Modulation Index (Deviation Ratio)

For FM/PM, the modulation index or deviation ratio ,β, is defined as $\beta = \Delta f / B_m$.
Accordingly we may rewrite Carson's rule as

$$BW_{FM} \approx 2[\Delta f + B_m] = 2B_m[\Delta f / B_m + 1] = 2B_m[\beta + 1] \quad \text{(Hz)}$$

For PM

Notice that the above results can be readily extended for PM:

$$\omega_i(t) = \omega_c + k_p \dot{m}(t)$$

$$\Delta\omega = k_p \dot{m}_p \quad \text{,where} \quad \dot{m}_p = \max[\dot{m}(t)]$$

$$BW_{PM} = 2(\Delta f + B_m) = 2\left(\frac{k_p \dot{m}_p}{2\pi} + B_m\right)$$

Notice that

- BW_{FM} depends on the max($m(t)$).
- BW_{PM} depends on the frequency content (change of $m(t)$) which is related to $\max[\dot{m}(t)]$.

Examples for Bandwidth Estimation

On the text book do examples 5.3-5.4 & 5.5

(In the class we start 5.3 then continue 5.3 & 5.4 to see the effect of doubling the amplitude of the message also examine the effect of time expansion/compression. Do example 5.5.

Historical Note

Read the historical Note about Edwin H. Armstrong (1890-1954)

Why FM?

We started FM with the objective of saving bandwidth but this out to be wrong. So why FM what are the features that make FM applicable?

1. FM can exchange bandwidth for quality
   - Signal to Noise Ration (SNR) α (Transmission bandwidth)^2
   - Recall that AM has limited bandwidth
2. FM has constant amplitude (Constant Power)
3. FM is Immune to nonlinearity