Trigonometric Fourier Series

- Importance of frequency representation
- Fourier series are usually used for periodic signals
- We can express a signal by a trigonometric Fourier series (FS) with period T_0

$$\boxed{g(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t)}$$

where

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t)dt$$

$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)cos(n\omega_0 t)dt$ n=1,2,3,….

$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)sin(n\omega_0 t)dt$ n=1,2,3,….

Compact Trigonometric Fourier Series

Since $a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t) = c_n cos(n\omega_0 t + \theta_n)$

where $c_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = tan^{-1}\left(\frac{-b_n}{a_n}\right)$, $c_0 = a_0$

$$\boxed{g(t) = c_0 + \sum_{n=1}^{\infty} c_n cos(n\omega_0 t + \theta_n)}$$

Note about the periodicity of the trigonometric Fourier Series

g(t) has to be periodic otherwise the F.S. ia only good for $t_1 \le t \le t_1 + T_0$

Amplitude Spectrum (single sided) is a plot c_n vs. ω,

Phase Spectrum (single sided) is a plot of θ_n vs. ω.

Exponential Fourier Series

Using Euler's formula $c_n cos(n\omega_0 t + \theta_n) = \frac{c_n}{2}\left[e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right]$

We may write

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}}$$

$$D_n = \frac{1}{T_0}\int_{T_0} g(t)e^{-jn\omega_o t}dt$$

Existence of the Fourier Series: Dirichlet conditions: $\int_{T_0} |g(t)|dt < \infty$

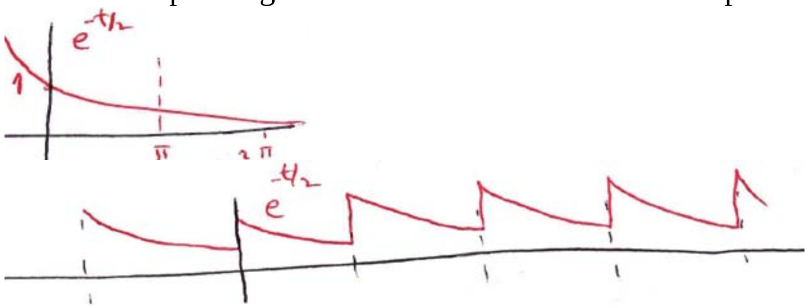
- The Fourier spectrum (Exponential Fourier Spectra or double sided spectra): plot of D_n magnitude or phase vs. ω. Remember that D_n is complex in general.
- Relation between the single sided and double sided spectra.

$\lvert D_n\rvert = \lvert D_{-n}\rvert = \frac{1}{2}c_n$, $D_0=c_0$, even functions
Angle(D_n)= θ_n and Angle(D_{-n})=$-\theta_n$, odd

(even magnitude +odd phase= complex conjugate symmetry)
We will be using the double sided spectra.

Example 2.7

Find the compact trigonometric Fourier Series for the exponential $e^{-t/2}$.



$$T_0 = \pi$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = 2$$

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(2nt) + b_n sin(2nt)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}dt = \frac{-2}{\pi}e^{-t/2}\Big|_0^{\pi} = \frac{-2}{\pi}[0.2079 - 1] = 0.504$$

Using Integration Tables (there is one in Appendix D, p774 , 3rd addition)

$$\int e^{ax} sin(bx)dt = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$$

$$\int e^{ax} cos(bx)dt = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}cos(2nt)dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}sin(2nt)dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

$$g(t) = 0.504\left[1 + \sum_{n=1}^{\infty}\frac{2}{1+16n^2}\left(cos(2nt) + 4n\, sin(2nt)\right)\right]$$

The compact form

$C_0 = a_0 = 0.504$

$C_n = \sqrt{a_n^2 + b_n^2} = 0.504\sqrt{\frac{4}{(1+16n^2)^2} + \frac{64n^2}{(1+16n^2)^2}} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$

$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1}(-4n) = -\tan^{-1}4n$

$g(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos(2nt - \tan^{-1}4n)$

$= 0.504 + 0.244 \cos(2t - 75.96^\circ) + \cdots$

n	0	1	2	3	4
C_n	0.504	0.244	0.125	0.084	0.063
θ_n	0	-75.96	-85.24	-86.42	-87.14

look at Exs 2.8 & 2.9

Ex. 2.10

Find the complex Fourier series

$T_0 = \pi$, $\omega_0 = \frac{2\pi}{T_0} = 2$

$\varphi(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$

$D_n = \frac{1}{T_0} \int_{T_0} \varphi(t) e^{-j2nt} dt = \frac{1}{\pi} \int_0^{\pi} e^{-t/2} e^{-j2nt} dt$

$= \frac{1}{\pi} \int_0^{\pi} e^{-(\frac{1}{2}+j2n)t} dt = \frac{-1}{\pi(\frac{1}{2}+j2n)} e^{-(\frac{1}{2}+j2n)t} \Big|_0^{\pi}$

$\varphi(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1+j4n} e^{j2nt}$

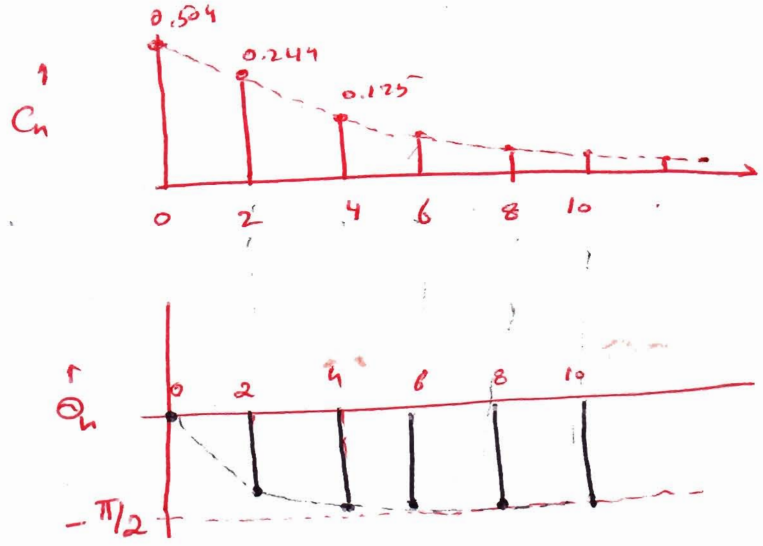
$= 0.504 \left[1 + \frac{1}{1+j4} e^{j2t} + \cdots \right.$

$\left. + \frac{1}{1-j4} e^{-j2t} + \cdots \right]$

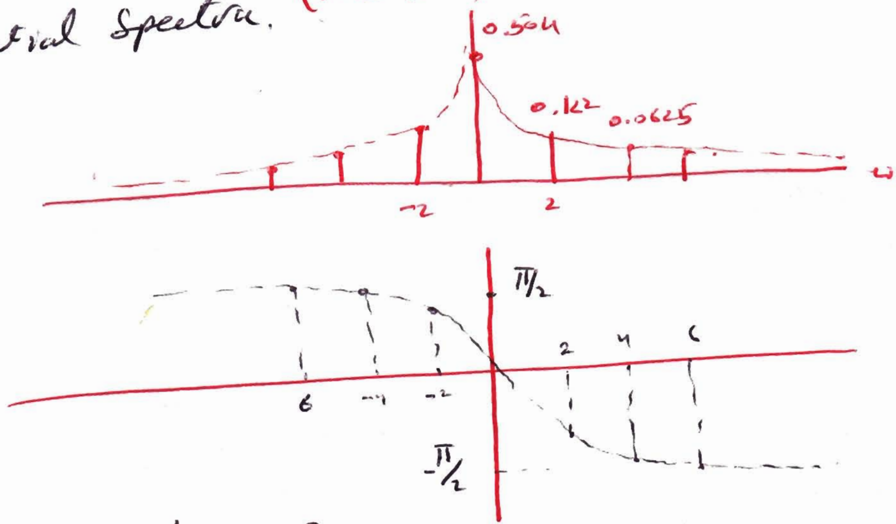
D_n complex D_n & D_{-n} are complex conjugates

Spectrum.

Trigonometric.



Exponential Spectra. (real $g(t)$)



What is a negative frequency?

just a way of representation indicating that there is a component at $(-n)$

Parsavals theorem.

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

(orthogonal add up).

$$P_g = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

$$g(t) = D_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} D_n e^{jn\omega_0 t}$$

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2 = D_0^2 + 2\sum_{n=1}^{\infty} |D_n|^2$$

for real $g(t)$ $|D_{-n}| = |D_n|$