King Fahd University of Petroleum & Minerals
Electrical Engineering Department
EE315: Probabilistic Methods in Electrical Engineering (112)

Major Exam II

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April 25, 2012
7:00-8:30 PM
Building 59-Rooms 2001-2004

Name: _________Key_________________

ID#______________________________

Question	Mark
1	/12
2	/8
3	/10
4	/10
Total	/40

Instructions:
1. This is a closed-books/notes exam.
2. The duration of this exam is one and half hours.
3. Read the questions carefully. Plan which question to start with.
4. CLEARLY LABEL ALL SIGNIFICANT VALUES ON BOTH AXIES OF ANY SKETCH
5. Work in your own and show the steps.
6. Strictly no mobile phones are allowed.

Good luck

Mark	Sec	Timing	Instructor
	1	SMW 9:00	Dr. Ahmed Masoud
	2	UT 10:00	Dr. Ali Muqaibel (Coordinator)
	3	UT 08:30	Dr. Saad Al-Ubaidi
	4	UT 10:00	Dr. Saad Al-Ubaidi

<u>Problem 1:</u> (12 points)

The joint pdf of random variables X & Y is given by $f_{X,Y}(x,y) = \begin{cases} k & 0 \leq y \leq x \leq 1 \\ 0 & otherwise \end{cases}$

Hint: carefully notice the region on which the joint pdf is non-zero

a. What is the value of k such that $f_{X,Y}(x,y)$ is a valid joint pdf.

Solution next page

b. Find $f_X(x)$ and $f_Y(y)$.

c. Find $P\left(0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}\right)$.

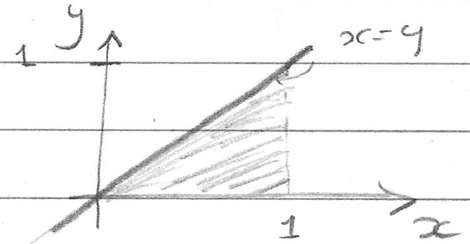
d. Find the conditional probability $f_X(x|y)$ and $f_Y(y|x)$.

e. Find the expected value, $E[Y|X]$, and the variance , $Var[Y|X]$.

①

$$f_{XY}(x,y) = \begin{cases} k & 0<y\leq x<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) The shaded area shown in the figure below is the region in the xy plane where $\{0<y\leq x<1\}$ which is equivalent to the region of $\{(0<x<1)\cap(0<y<x)\}$



$f_{XY}(x,y)$ should be integrated over the shaded region and equated to $\underline{\underline{1}}$ as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\Rightarrow \int_0^1\int_0^x k \cdot dy\, dx = k\int_0^1 ky\Big]_0^x = k\int_0^1 x\, dx$$

$$= k\left(\tfrac{1}{2}\right)$$

$$\Rightarrow \boxed{k=2}$$

(b) $$f_X(x) = \begin{cases} \int_0^x f_{XY}(x,y)\, dy = 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_y^1 f_{XY}(x,y)\, dx = 2(1-y) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

① Cont.

(c) $P(0<X<\frac{1}{2},\ 0<Y<\frac{1}{2}) = P(0<X<\frac{1}{2},\ 0<Y<X)$

$$= \int_0^{1/2}\int_0^{x} 2\,dy\,dx = \int_0^{1/2} 2y\Big]_0^{x} dx = x^2\Big]_0^{1/2} = \frac{1}{4}$$

(d) $$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{2}{2(1-y)} = \frac{1}{1-y} & 0<y\leq x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \frac{2}{2x} = \frac{1}{x} & 0<y\leq x<1 \\ 0 & \text{otherwise} \end{cases}$$

(e) $$E[Y/x] = \int_{-\infty}^{\infty} y\, f_{Y/X}(y/x)\, dx$$

$$= \int_0^{x} y\left(\frac{1}{x}\right) dy = \frac{y^2}{2x}\Bigg|_{y=0}^{x} = \frac{x}{2},\ 0<x<1$$

$$VAR[Y/x] = E[Y^2/x] - \left(E[Y/x]\right)^2$$

$$E[Y^2/x] = \int_0^{x} y^2\left(\frac{1}{x}\right) dy = \frac{y^3}{3x}\Bigg|_{y=0}^{x}$$

$$= \frac{x^2}{3}\ ,\ 0<x<1$$

$$\Rightarrow VAR[Y/x] = \frac{x^2}{3} - \left(\frac{x}{2}\right)^2 = \frac{x^2}{12} \quad 0<x\ 1$$

<u>Problem 2:</u> (6+2= 8 points)

X and Y are jointly Gaussian random variables, with joint pdf: $f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}$

They are transformed to two new random variables M and N according to

$$M = \frac{1}{\sqrt{2}}(Y+X) \qquad , \qquad N = \frac{1}{\sqrt{2}}(Y-X)$$

We would like to know if the new random variables are independent or not. *Show your work.*

By comparing the given joint pdf and the general form . Both X and Y have zero mean and unit variance

$$E[MN] = E\left[\frac{1}{\sqrt{2}}(Y+X)\frac{1}{\sqrt{2}}(Y-X)\right] = \frac{1}{2}E[X^2 - Y^2] = 0$$

Uncorrelated Gaussian means independent.

Consider two jointly Gaussian random variables X and Y with zero mean and covariance matrix $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix}$, compute the correlation coefficient.

$$\boldsymbol{\rho = \frac{C_{XY}}{\sigma_X \sigma_Y} = \frac{\frac{1}{2}}{\sqrt{1}\sqrt{2}} = \frac{1}{2\sqrt{2}}}$$

Problem 3: **(5+5=10 points)**

Consider a random variable X with k as a parameter and a moment generating function (MGF) given by

$$M_X(v) = (1-2v)^{-\frac{k}{2}}$$

Find the variance of X in terms of k.

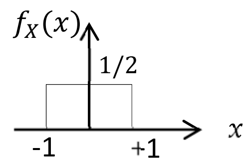
$$m_n = \frac{d^n M_X(v)}{dv^n}\Big|_{v=0}$$

$$m_1 = k$$

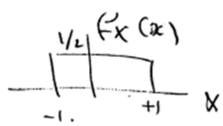
$$m_2 = k^2 + 2k$$

$$\text{Variance}=m_2 - m_1^2 = k^2 + 2k - (k)^2 = 2k$$

The probability density function (pdf) of a random variable X is shown in the figure. Find $f_Y(y)$ where $= 1 - X^2$.



$Y = 1 - X^2$



Note $0 \leq Y \leq 1$

$F_Y(y) = Pr(Y \leq y)$

$= Pr(1 - X^2 \leq y)$

$\therefore F_Y(y) = 1 \qquad 1 \leq y$

$F_Y(y) = 0 \qquad y \leq 0$

$\Rightarrow 0 \leq y \leq 1 \qquad F_Y(y) = Pr(Y \leq y) =$

$= Pr(-c \leq X \leq c)$

$c = \sqrt{1-y} \qquad = 1 - [Pr(X \leq c) - Pr(X \leq -c)]$

$\therefore F_Y(y) = 1 - F_X(\sqrt{1-y}) + F_X(-\sqrt{1-y})$.

$$\therefore f_Y(y) = \begin{cases} 1 & y \geq 1 \\ F_X(\sqrt{1-y}) - F_X(-\sqrt{1-y}) & 0 \leq y \leq 1 \\ 0 & y \leq 0 \end{cases}$$

$$\therefore f_Y(y) = \begin{cases} 0 & y \geq 1 \\ -\frac{d}{dy}\left(F_X(\sqrt{1-y}) - F_X(-\sqrt{1-y})\right) & 0 \leq y \leq 1 \\ 0 & y < 0 \end{cases} = \begin{cases} 0 & 1 \leq y \\ \frac{1}{2}\frac{1}{\sqrt{1-y}} & 0 \leq y \leq 1 \\ 0 & y < 0 \end{cases}$$

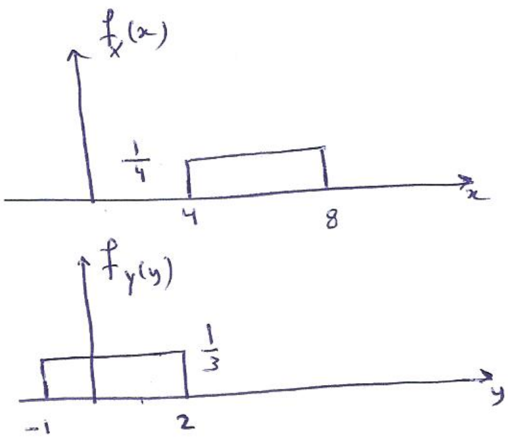
itali.

Problem 4: **(7+3=10 points)**

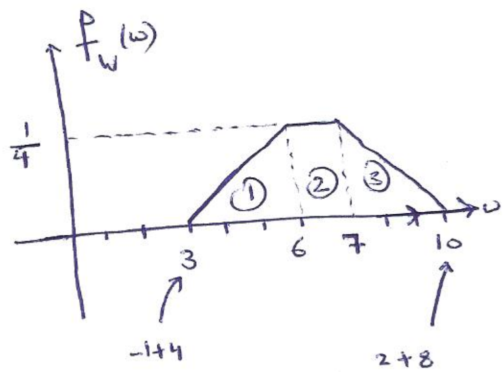
Find and sketch the density of $W = X + Y$ where the densities of X and Y are assumed to be

$$f_X(x) = \frac{1}{4}[u(x-4) - u(x-8)]$$

$$f_Y(y) = \frac{1}{3}[u(y+1) - u(y-2)]$$



Convolution

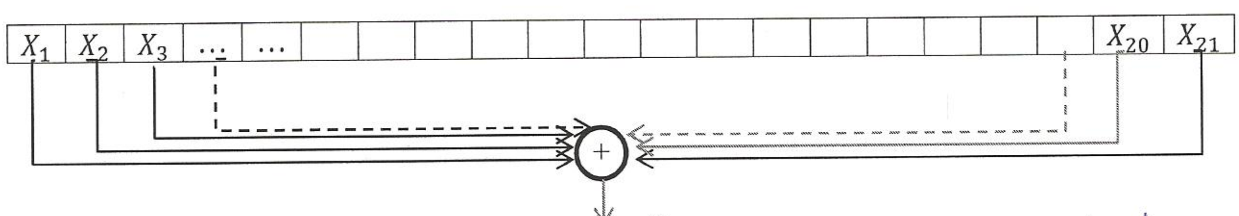


Convolution full overlap $= \frac{1}{3} \cdot \frac{1}{4}(3) = \frac{1}{4}$

Check area. ① ② ③

$\frac{3}{2}\frac{1}{4} + 1\,\frac{1}{4} + \frac{3}{2}\frac{1}{4} = \frac{3}{8} + \frac{2}{8} + \frac{3}{8} = 1$ ✓

Let X_i be independent identically distributed random variables with zero mean and unit variance. The random variable Z is generated by summing all the 21 random variables X_i as shown in the figure. Find an approximate expression for $f_Z(z)$.



Using central limit theorem $f_Z(3)$ can be approximated by Gaussian pdf

$E[Z] = E[X_1] + \dots E[X_{21}] = 0 = a_3$

$Var[Z] = \sum_{i=1}^{21} \sigma^2_{X_i} = 21(1) = 21 = \sigma_3^2$

$f_Z(3) \simeq \frac{1}{\sqrt{2\pi\sigma_Z^2}} e^{-(3-a_3)^2/2\sigma_3^2} \simeq \frac{1}{\sqrt{2\pi(21)}} e^{-3^2/42}$