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The *z*-transform
Part 1

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The material to be covered in this lecture is as follows:

- Introduction to the *z*-transform
- Definition of the *z*-transform
- Derivation of the *z*-transform
- Region of convergence for the transform
- Examples.

After finishing this lecture you should be able to:

- Find the *z*-transform for a given signal utilizing the *z*-transform definition
- Calculate the region of convergence for the transform

Derivation of the z-Transform

- The z-transform is the basic tool for the analysis and synthesis of discrete-time systems.
- The z-transform is defined as follows:
- $$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$
- The coefficient $x(nT_s)$ denote the sample value and z^{-n} denotes that the sample occurs n sample periods after the $t = 0$ reference.
- Note that the lower limit of the summation can start from zero if the signal is causal (Unilateral z-transform)
- Rather than starting form the given definition for the z-transform, we may start from the continuous-time function and derive the z-transform. This is done in the next slide.

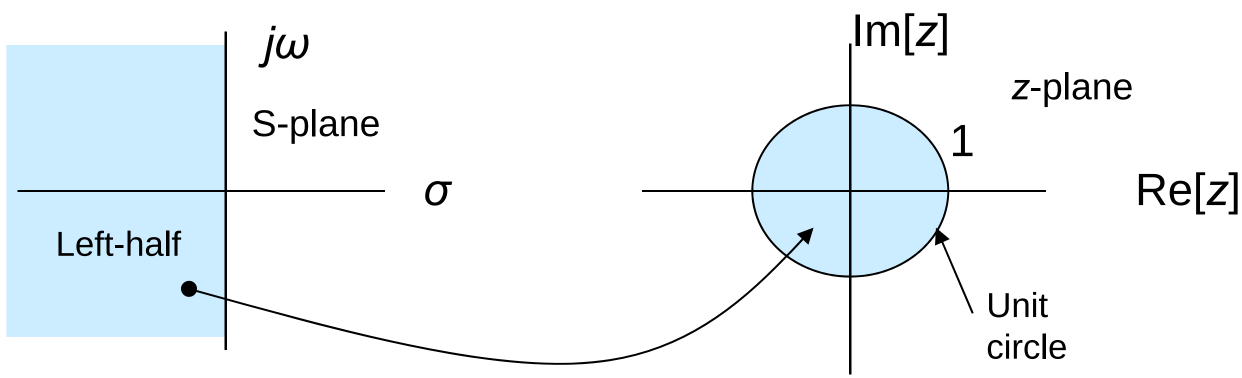
Derivation of the z-transform

- The sampled signal may be written as $$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$
- Since $\delta(t - nT_s) = 0$ for all t except at $t = nT_s$, $x(t)$ can be replaced by $x(nT_s)$. $$x_s(t) = \sum_{n=0}^{\infty} x(nT_s)\delta(t - nT_s)$$
  - And Assuming $x(t) = 0$ for $t < 0$. Then,
- Taking Laplace transform yields $$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nt)\delta(t - nT)e^{-st}dt$$
- Rearranging $$X_s(s) = \sum_{n=0}^{\infty} x(nT)\int_0^{\infty} \delta(t - nT)e^{-st}dt$$
- By sifting property of the delta function

$$X_s(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT}$$

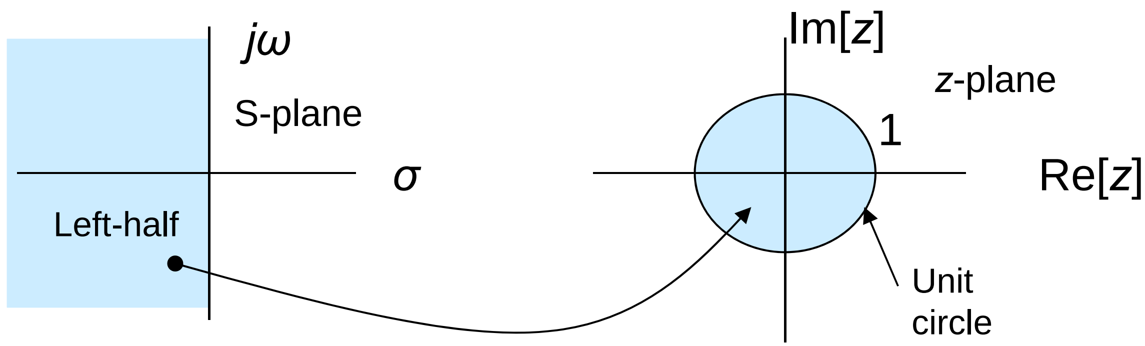
Continue Derivation…

- Defining the complex variable *z* as the Laplace time-shift operator $z = e^{sT}$
- $X(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT}$ becomes, $X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}$
- We could have started from the last expression but it is good to relate to the s-domain
- In the s-domain the left-half plane corresponds to $\sigma < 0$ is mapped to $|z| < 1$ in the z-plane which is the region inside the unit circle.



Region of Convergence (ROC)

- $z = e^{sT}$
- $s = \sigma + j\omega$
- $z = e^{\sigma T} e^{j\omega T}$
- $|z| = e^{\sigma T}$



- $|z|$ is converged for $\sigma < 0$ (left-half of *s*-plane). This corresponds to $|z| < 1$. This is the region inside the unit circle.
- $|z|$ is NOT converged for $\sigma > 0$ (right-half of *s*-plane). This corresponds to $|z| > 1$ which is the region outside the unit circle
- The mapping of the Laplace variable s into the *z*-plane through $z = e^{sT}$ is illustrated in the figure.

The Z-Transform in Summary

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Where $z = e^{sT}$ and $n >= 0$

- The coefficient $x(nT)$ denotes the sampled value
- The square bracket is used to indicate discrete times.
- z^{-n} denotes that the sample occurs n sample periods after the $t = 0$ reference.
- e^{sT} is simply the T-second time shift
- The parameter z is simply shorthand notation for the Laplace time shift operator
- For instance, $30z^{-40}$ denotes a sample, having value 30, which occurs *40* sample periods after the t=0 reference
- Matlab has special tools for z-transform: *ztrans*, *iztrans* , *pretty*

Example 1:

- Determine the *z*-transform for the following signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & otherwise \end{cases}$$

- Solution:
- We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

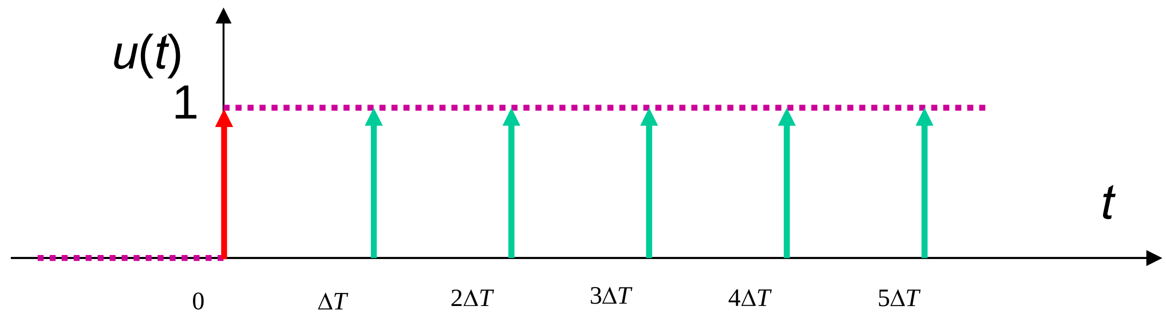
- hence

$$X(z) = \sum_{-1}^{2} x[n] z^{-n} = x[-1] z^{-(-1)} + x[0] z^{-(0)} + x[1] z^{-(1)} + x[2] z^{-(2)}$$

$$X(z) = z + 2 - z^{-1} + z^{-2}$$

Example 2: Sampled Step Function (*Important Functions*)

- Consider a unit step sample sequence defined by $x[n] = 1, n \geq 0$
- Find the z-transform.



- Solution

$$U(z) = X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots\dots\dots = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

- The sum converges absolutely to $\frac{1}{1-z^{-1}}$ outside the unit circle $|z| > 1$

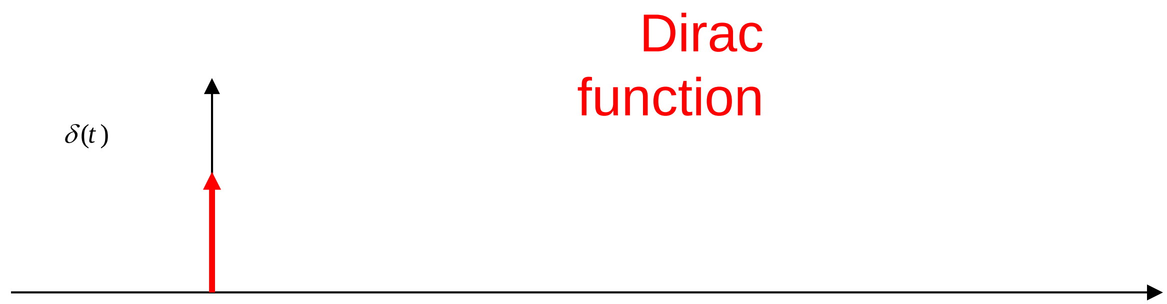
$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

Sampled Dirac Delta Function (*an other important function*)

- The Dirac Delta Function is defined to be
- For a delayed version of delta is defined as
- Applying the definition of the z-transform

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$



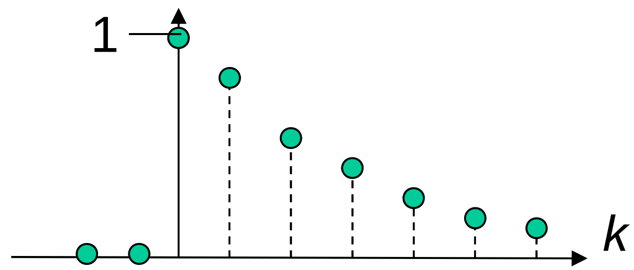
$$X(z) = \sum_{k=0}^{\infty} \delta(t) z^{-s\Delta T} = \delta(0) = 1$$

$$X(z) = 1$$

The Unit Exponential Sequence

- The unit exponential sequence is defined to be

$$x[k] = \begin{cases} e^{-\alpha k} & k, \alpha > 0 \\ 0 & k < 0 \end{cases}$$



- Apply z-transform definition $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$, we get

$$X(z) = \sum_{k=0}^{\infty} e^{-\alpha k} z^{-k} = \sum_{k=0}^{\infty} (e^{-\alpha} z^{-1})^k \qquad where\, |z| > e^{-\alpha}$$

$$X(z) = \frac{1}{1 - e^{-\alpha} z^{-1}} = \frac{z}{z - e^{-\alpha}}$$

$$X[k] = \frac{1}{1 - e^{-\alpha} z^{-1}} = \frac{z}{z - e^{-\alpha}}$$

$$if \;\; k = e^{-\alpha} \;\; then \qquad X(z) = \frac{1}{1 - kz^{-1}} = \frac{z}{z - k}$$

Example 3 with Poles and Zeros

- Determine the z-transform of the signal
- $$x[n] = 0.5^n u[n]$$
- Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane
- Solution:
- Substituting is the definition of the z-transform

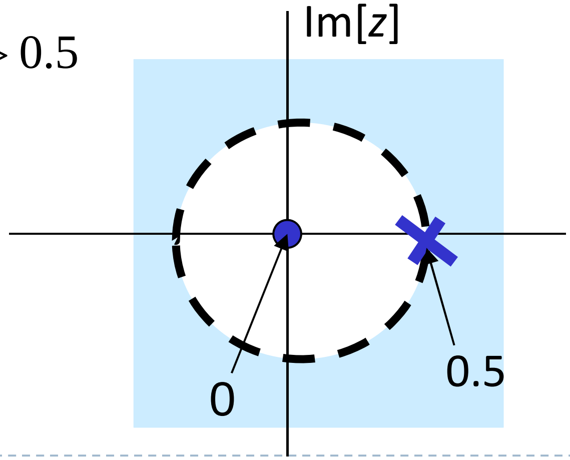
$$X(z)=\left(\frac{1}{2}\right)^0 z^{-0}+\left(\frac{1}{2}\right)^1 z^{-1}+\left(\frac{1}{2}\right)^2 z^{-2}+\ldots$$

$$X(z)=1+\left(\frac{1}{2z}\right)+\left(\frac{1}{4z^2}\right)+\ldots$$

- $X(z) = \sum_{n=-\infty}^{\infty} 0.5^n u[n] z^{-n} = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \sum_{n=0}^{\infty} (\frac{0.5}{z})^n$
- This is a geometric series of infinite length in the ratio 0.5/z; the sum converges, provided that $|\frac{0.5}{z}| < 1$ or $|z| > 0.5$. Hence the z-transform is

$$X(z)=\sum_{n=0}^{\infty}\left(\frac{0.5}{z}\right)^n=\frac{1}{1-0.5z^{-1}}, \qquad |z|>0.5$$

$$=\frac{z}{z-0.5}, \qquad |z|>0.5$$



- ***Pole at $z = 0$, zero at $z = 0.5$,***
- ***ROC is the light blue region***

Self Test 1:

- Find the *z*- transform of the following signal:

$$X(nT) = a^n \cos\left(\frac{n\pi}{2}\right)$$

- Hint : $\cos\left(\frac{n\pi}{2}\right) = 0\ for\ n\ odd\ and\ \pm 1\ for\ even\ n$
- Answer:

$$X(z) = \frac{1}{1 + a^2 z^{-2}} \qquad |z| > |a|$$

Self Test 2:

- Determine the z-transform of the signal

$$x[n] = -u[-n-1] + 0.5^n u[n]$$

- Depict the ROC and the locations of poles and zeros of $X(z)$ in the z-plane
- Answer:

- the sum converges, provided that $|z| > 0.5$ and $|z| < 1$.

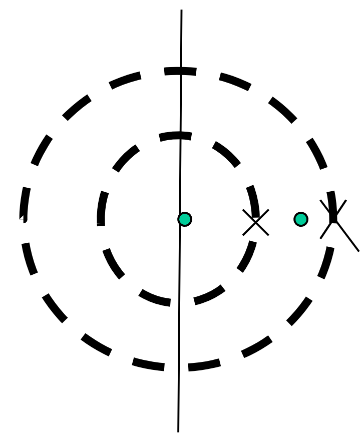
$$X(z) = \sum_{n=0}^{\infty}\left(\frac{0.5}{z}\right)^n - \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{0.5}{z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k$$

$$X(z) = \frac{1}{1-0.5z^{-1}} + 1 - \frac{1}{1-z}, \qquad 0.5 < |z| < 1$$

$$= \frac{z(2z-1.5)}{(z-0.5)(z-1)}, \qquad 0.5 < |z| < 1$$

Poles at z=0.5, 1, zeros at z=0, 0.75. ROC is the region in between



Continue Self-test

$$X(z) = \frac{1}{1-0.5z^{-1}} + 1 - \frac{1}{1-z}, \qquad 0.5 < |z| < 1$$

$$= \frac{z(2z-1.5)}{(z-0.5)(z-1)}, \qquad 0.5 < |z| < 1$$

Poles at z=0.5, 1, zeros at z=0, 0.75. ROC is the region in between