Introduction to Discrete-Time Signals and Systems

z-Transform and Applications to Discrete-Time Systems

Lecture #40

The material to be covered in this lecture is as follows:

- Properties of the *z*-transform
  - Linearity
  - Initial and final value theorems
  - Time-delay
- *z*-transform table
- Inverse *z*-transform
- Application of *z*-transform to discrete-time systems

After finishing this lecture you should be able to:

- Find the z-transform for a given signal utilizing the z-transform tables
- Utilize the z-transform properties like the initial and final value theorems
- Find the inverse z-transform.
- Utilize z-transform to perform convolution for discrete-time systems.

Derivation of the *z*-Transform

- The *z*-transform is defined as follows:

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(nT)\, z^{-n}}$$

- The coefficient $x(nT)$ denote the sample value and z^{-n} denotes that the sample occurs *n* sample periods after the *t*=0 reference.
- Rather than starting form the given definition for the *z*-transform, we may build a table for the popular signals and another table for the *z*-transform properties.
- Like the Fourier and Laplace transform, we have two options either to start from the definition or we may utilize the tables to find the proper transform.
- The next slide illustrates a few *z*-transform pairs.
- Then we will investigate some of the *z*-transform properties:
    - Linearity
    - Time-shifting property
    - Initial and final value theorems

Table of z-transform pairs

$f(t)$	$F(z)$
1	1
$\delta(t-k\Delta T)$	z^{-k}
$u(t)$	$\dfrac{z}{z-1}$
t	$\dfrac{\Delta T z}{(z-1)^2}$
t^2	$\dfrac{\Delta T^2 z(z+1)}{(z-1)^3}$
e^{-at}	$\dfrac{z}{z-e^{-a\Delta T}}$
te^{-at}	$\dfrac{\Delta T z e^{-a\Delta T}}{(z-e^{-a\Delta T})^2}$
$a^n u[n]$	$\dfrac{z}{z-a}$

Linearity of the z-Transform

If

$x_1[n] \overset{Z}{\leftrightarrow} X_1(z)$ with region of convergence, $ROC=R_1$.

and

$x_2[n] \overset{Z}{\leftrightarrow} X_2(z)$ with region of convergence, $ROC=R_2$.

Then

$$ax_1[n]+bx_2[n] \overset{Z}{\leftrightarrow} aX_1(z)+bX_2(z) \quad \text{with } ROC = R_1 \cap R_2$$

- This follows directly from the definition of the z-transform because the summation operator is linear.
- It is easily extended to a linear combination of an arbitrary number of signals.
- This property includes the multiplication by constant property which states that if the signal is scaled by a constant its z-transform will be scaled by the same constant.

$$ax_1[n] \overset{Z}{\leftrightarrow} aX_1(z)$$

Time-Shifting property for the *z*-Transform

If

$$x[n] \overset{Z}{\leftrightarrow} X(z) \quad ROC{=}R$$

Then

$$x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0} X(z) \quad ROC{=}R$$

Proof

$$Z\{x[n-1]\} = \sum_{n=-\infty}^{\infty} x[n-1] z^{-n}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x[n-1] z^{-(n-1)}$$

$$= z^{-1} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-1} Z\{x[n]\}$$

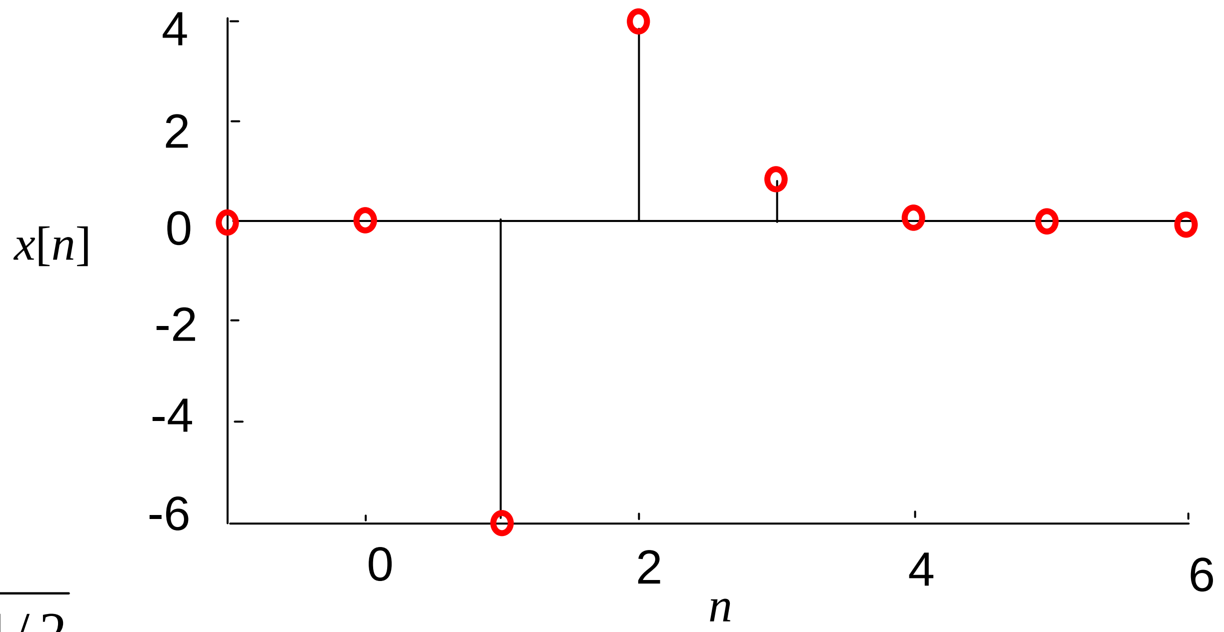
This property will be very important for producing the ***z*-transform transfer function of a difference equation** which uses the property:

$$x[n-1] \overset{Z}{\leftrightarrow} z^{-1} X(z)$$

Example 40.1: Properties of the *z*-transform

Find the *z*-transform for the input signal

$$x[n] = 7(1/3)^{n-2}u[n-2] - 6(1/2)^{n-1}u[n-1]$$



Solution:

We know that

$$a^n u[n] \overset{Z}{\leftrightarrow} \frac{z}{z-a}$$

So

$$X(z) = 7z^{-2}\frac{z}{z-1/3} - 6z^{-1}\frac{z}{z-1/2}$$

$$= 7\frac{1}{z^2 - 1/3z} - 6\frac{1}{z-1/2}$$

Initial and Final Value Theorems

If $x[n]$ has a z-transform $X(z)$ and if lim $X(z)$ as $z \to \infty$ exists, then

$$\boxed{\lim_{n \to 0} x[n] = x[0] = \lim_{z \to \infty} X(z)}$$

This theorem can be easily proven by the definition of the z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \ldots$$

As we take the limit all terms will be zero except the first term

The final value theorem which is given by

$$\boxed{\lim_{n \to \infty} x[n] = \lim_{z \to 1} \left(\left(1 - z^{-1} \right) X(z) \right)}$$

Example 40.2: Application of the initial and final value theorems

Find the initial and final values for the following signal expressed in its *z*-transform

$$F(z) = \frac{0.792z^2}{(z-1)(z^2 - 0.416z + 0.208)}$$

Solution:

Initial-value $$F(z \to \infty) = \frac{0.792z^2}{z^3} = 0$$

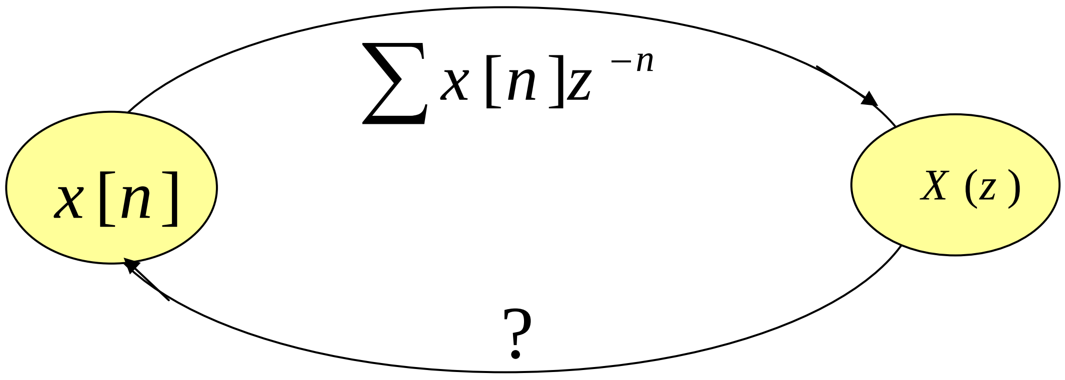
Final –value $$f(n \to \infty) = \frac{0.792}{(1 - 0.416 + 0.208)} = 1$$

These answers can be justified by looking at the expansion of the given expression

$$F(z) = 0.792z^{-1} + 1.12z^{-2} + 1.091z^{-3} + 1.01z^{-4} + 0.983z^{-5} + 0.989z^{-6} + 0.99z^{-7}....$$

- The coefficient for z^0 is zero which is the initial value.
- The coefficient converges to one as the negative power of *z* increases which corresponds to the final value.

Inverse *z*-transform



- The inverse operation for the z-transform my be accomplished by:
  - Long division
  - Partial fraction expansion
- The z-transform of a sample sequence can be written as

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots.$$

- If we can write $X(z)$ into this form, the sample values can be determined by inspection.
- When $X(z)$ is represented in a ratio of polynomials in z, this can be easily achieved by long division.
- Before carrying out the division, it is convenient to arrange both the numerator and the denominator in ascending powers of z^{-1}.

Inverse z-transform using Partial Fraction Expansion

- Alternatively, we may avoid the long division by partial fraction expansion. The idea is similar to the method used for inverse Laplace transform.
- The objective is to manipulate $X(z)$ into a form that can be inverse z-transformed by using z-transform tables.
- Because of the forms of transforms,
  - it is usually best to perform partial fraction expansion of $H(z)/z$.
  - As an alternative z^{-1} can be treated as the variable in the partial fraction expansion.
- Important: before doing partial-fraction expansion, make sure the z-transform is in proper rational function of z^{-1} !

Example 40.3: Inverse *z*-Transform Using Partial Fraction Expansion

Find the inverse z-transform using both partial fraction expansion and long division

$$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

Solution:

If we treat z^{-1} as the variable in the partial fraction expansion, we can write

$$X(z) = \frac{1}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

Utilizing Heaviside's Expansion Method:

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

(1) $(1-z^{-1})X(z) = \dfrac{1}{1-0.2z^{-1}} = A + \dfrac{B(1-z^{-1})}{1-0.2z^{-1}} \overset{z=1}{\Rightarrow}$

$$\frac{1}{1-0.2} = A + \frac{B\cdot 0}{1-0.2} \Rightarrow A = \frac{1}{0.8} = 1.25$$

(2) $(1-0.2z^{-1})X(z) = \dfrac{A(1-0.2z^{-1})}{1-z^{-1}} + \dfrac{B\cdot(1-0.2z^{-1})}{1-0.2z^{-1}} \Rightarrow \dfrac{1}{1-z^{-1}} = \dfrac{A(1-0.2z^{-1})}{1-z^{-1}} + B$

Continue ...Example 40.3

$$\xRightarrow{1-0.2z^{-1}=0\ (z=0.2)} \qquad B = 1/(1-5) = -1/4 = -0.25 \quad \rightarrow$$

$$X(z) = \frac{1.25}{1-z^{-1}} + \frac{-0.25}{1-0.2z^{-1}} \Rightarrow x(nT) = 1.25 - 0.25(0.2)^n$$

From which we may find that $x(0)=1$, $x(T)=1.2$, $x(2T)=1.24$, $x(3T)=1.248$

We may get to the same answer using long division. $X(z)$ is written as

$$X(z) = \frac{z^2}{z^2 - 1.2z + 0.2}$$

which is, after multiplying numerator and denominator by z^{-2}

$$X(z) = \frac{1}{1 - 1.2z^{-1} + 0.2z^{-2}}$$

Now, it is left for you to show that the long division will result in the same answer given by

$$X(z) = 1 + 1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3} + \ldots.$$

Discrete-Time Systems

- For continuous-time systems, differential equation may solved using Laplace transform
- Similarly discrete-time systems result in ***Difference Equations*** which may be solved using z-transform
- Recall that discrete-time systems process a discrete-time input signal to produce a discrete-time output signal.
- The general symbolic notation for Discrete-Time System:

$$y(nT) = H[\, x(nT) \,]$$

- Similar to continuous-time systems we may define some properties for the discrete-time systems. For example,

1.Shift-invariant system: a system is shift invariant if $H[x(nT - n_0T)] = y(nT - n_0T)$ true for any n_0 . (Similar to the concept of time-invariant systems for continuous-time)
2. Causal and noncausal systems: physical Description: A system is causal or nonanticipatory if the system's response to an input does not depend on future values of the input.
3. Linear System
Linear System $\Leftrightarrow H[\alpha_1 x_1(nT) + \alpha_2 x_2(nT)] = \alpha_1 H[x_1(nT)] + \alpha_2 H[x_2(nT)]$

Application of the z-Transform to Linear Discrete-Time Signals

Linear Systems: can be modeled as

$$y(nT)=\sum_{k=-\infty}^{\infty}x(kT)h(nT-kT) \quad \text{or} \quad y(nT)=\sum_{k=-\infty}^{\infty}h(kT)x(nT-kT)$$

Convolution

$h(kT)$: response of the shift-invariant linear system at $t=kT$ to an impulse input applied at $t=0$.

Causal systems: $h(kT)=0 \quad \forall k<0$

If the system is linear and causal in addition to the fact that $x(kT)\equiv 0 \qquad (k<0)$ then

$$\Rightarrow y(nT)=\sum_{k=-\infty}^{\infty}x(kT)h(nT-kT)$$

$$\overset{x(kT)=0(k<0)}{=}\sum_{k=0}^{\infty}x(kT)h(nT-kT)$$

$$(nT-kT\geq 0)\; nT\geq kT \rightarrow \overset{causal}{=}\sum_{k=0}^{n}x(kT)h(nT-kT)=\sum_{k=0}^{n}h(kT)x(nT-kT)$$

Transfer Function in the z-Domain

- The z-transform is **linear**
- There is a simple relationship for a signal **time-shift**

$$x[n-1] \overset{z}{\leftrightarrow} z^{-1}X(z)$$

- This is fundamental for deriving the transfer function of a difference equation which is expressed in terms of the input-output signal delays
- The **transfer function** of a discrete time LTI system is the z-transform of the system's impulse response
- The transfer function is a rational polynomial in the complex number z.
- Convolution is expressed as multiplication

$$x[n]*h[n] \overset{z}{\leftrightarrow} X(z)H(z)$$

and this can be solved for particular signals and systems

Example 40.4: Discrete-Time Convolution

Calculate the output of a first order difference equation of a input signal $x[n] = 0.5nu[n]$

$$0.5^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{z}{z-0.5}$$

System transfer function (z-transform of the impulse response)

$$y[n] - 0.8y[n-1] = x[n]$$

Taking the *z*-transform of the difference equation

$$Y(z) - 0.8z^{-1}Y(z) = X(z)$$

$$Y(z)(1-0.8z^{-1}) = X(z)$$

$$H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$$

The (*z*-transform of the) output is therefore the product:

$$Y(z) = \frac{z^2}{(z-0.5)(z-0.8)}$$

$$= \frac{1}{0.3}\left(\frac{0.8z}{(z-0.5)} - \frac{0.5z}{(z-0.8)}\right)$$

ROC $|z|>0.8$

$$y[n] = (0.8 * 0.5^n u[n] - 0.5 * 0.8^n u[n]) / 0.3$$

Self Test

Question 1:

If $Z(x(nT)) = \dfrac{1}{1+0.5z^{-1}}$, what's $Z(x(nT-2T))$?

Answer: $\dfrac{z^{-2}}{1+0.5z^{-1}}$

Question 2:

Find the z-transform for $Y(z) = \dfrac{1}{z^2 - 1.2z + 0.2}$

Answer: $y(nT) = 5\delta(n) + 1.25 - 6.25(0.2)^n$ (Try to find a relation with example 40.3)

Question 3:

$$x(nT) = \left(\frac{1}{4}\right)^n u(n-3) = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} u(n-3)$$

$$h(nT) = \left(\frac{1}{3}\right)^n u(n-5) = \left(\frac{1}{3}\right)^5 \left(\frac{1}{3}\right)^{n-5} u(n-5)$$

Find $x(nT)*h(nT)$

Answer : $x(nT)*h(nT) = (\frac{1}{4})^3(\frac{1}{3})^5[4(\frac{1}{3})^{n-8} - 3(\frac{1}{4})^{n-8})]u(n-8)$

$$x(t)$$

Question 4 :

Calculate the step response to the system describe by the following difference equation

$$6y[n]-5y[n-1]+1y[n-2]=x[n]$$

Answer

$$u[n]\overset{z}{\leftrightarrow}X(z)=\frac{1}{1-z^{-1}}$$

$$H(z)=\frac{1}{6-5z^{-1}+1z^{-2}}=\frac{1}{(2-z^{-1})(3-z^{-1})}$$

$$Y(z)=\frac{1}{(3-z^{-1})(2-z^{-1})(1-z^{-1})}$$

$$=0.5\frac{1}{(3-z^{-1})}-\frac{1}{(2-z^{-1})}+0.5\frac{1}{(1-z^{-1})}$$

$$=0.167\frac{1}{(1-1/3z^{-1})}-0.5\frac{1}{(1-1/2z^{-1})}+0.5\frac{1}{(1-z^{-1})}$$

$$y[n]=(0.167(1/3)^n-0.5(1/2)^n+0.5)u[n]$$

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