Introduction to Discrete-Time Signals and Systems

Digital to Analog Conversion (Reconstruction) & Quantization Error

Lecture #38

The material to be covered in this lecture is as follows:

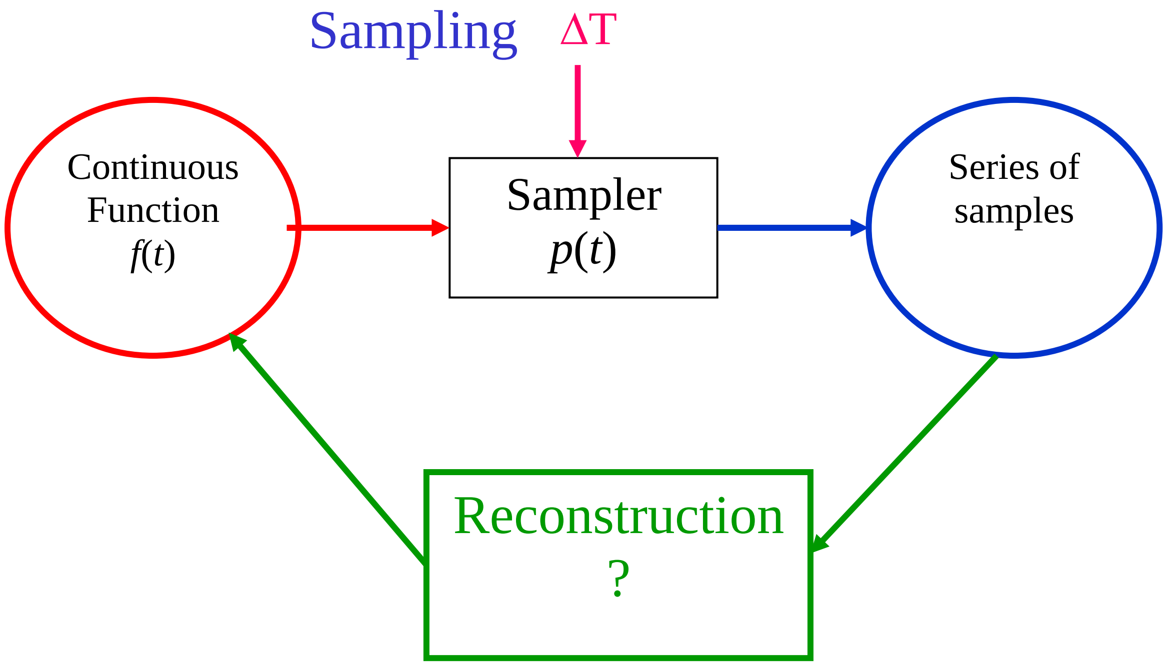
- Introduction to digital to analog (D/A) reconstruction
- Reconstruction/ Interpolation
    - Reconstruction in the frequency domain: Low-pass filter
    - Interpolation in the time domain: *sinc* function
    - Practical reconstruction (e.g. using RC circuit)
- Quantization Effect

After finishing this lecture you should be able to:

- Specify the different characteristics of the reconstruction low-pass filter.
- Perform reconstruction in the time-domain utilizing the *sinc* interpolation function.
- Understand the effect of practical reconstruction (e.g. using RC circuits)
- Describe the aliasing effects and the condition required to avoid aliasing.
- Evaluate the basic parameters of the quantizer and quantify their effect on the quantization noise.

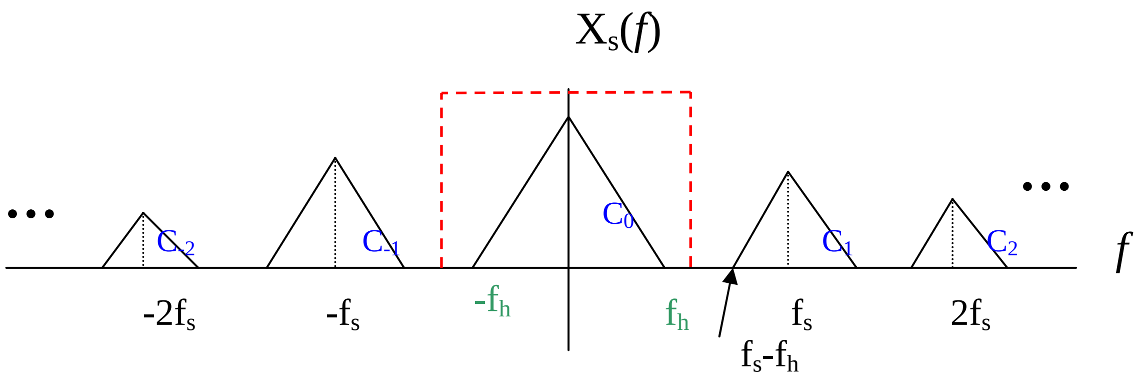
Introduction to signal reconstruction

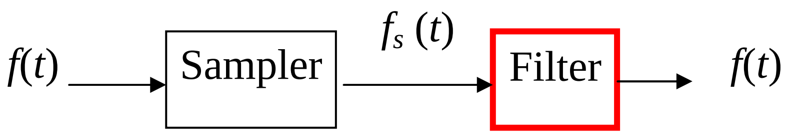
- The continuous signal can be reconstructed from its samples.
- Recall that analog to digital conversion is important as it allows us to process the signal in the digital domain. After the processing is over one has to reconstruct the continuous signal.
- Reconstruction is important as the final signal (audio/video/…) is usually required in its continuous form.



Reconstruction: Digital to Analog Conversion

Recall that the spectrum of the sampled continuous times-signal $x(t)$ is composed of the spectrum of $x(t)$ plus the spectrum of $x(t)$ translated to each harmonic of the sampling frequency.

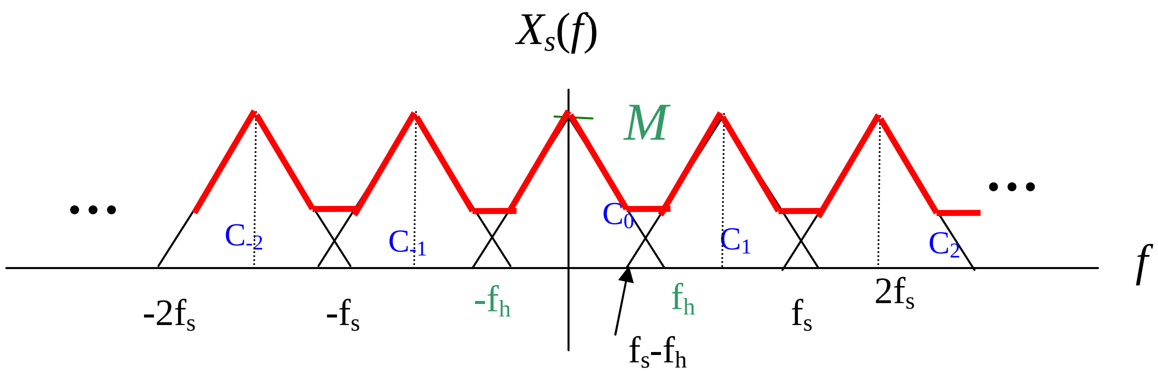




The original signal can be perfectly reconstructed using a low-pass filter with cut-off frequency equals to $f_s/2$ provided that the original signal was sampled at a frequency above $2f_h$.

Aliasing

- If the original signal was sampled at a rate less than twice the highest frequency then the translated spectrums will overlap and the original signal will **not** be reconstructed properly.
- This effect is known as aliasing and it is illustrated in the figure below



Ideal Reconstruction Filter

An ideal low-pass filter can be used to reconstruct the data. It has the following transfer function

$$H(f) = \begin{cases} T & |f| < 0.5f_s \\ 0 & otherwise \end{cases}$$

Using Inverse Fourier Transform

$$h(t) = T\int_{-f_s/2}^{f_s/2} e^{j2\pi ft}\,df \;= \frac{T}{j2\pi f}\left(e^{j2\pi f_s t} - e^{-j2\pi f_s t}\right)$$

$$h(t) = \frac{\sin\left(\pi f_s t\right)}{\pi f_s t} = \mathrm{sinc}\left(f_s t\right)$$

Note: that the impulse response is not time limited and non-causal.

Using the convolutional Integral, we may write the constructed signal as

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right)$$

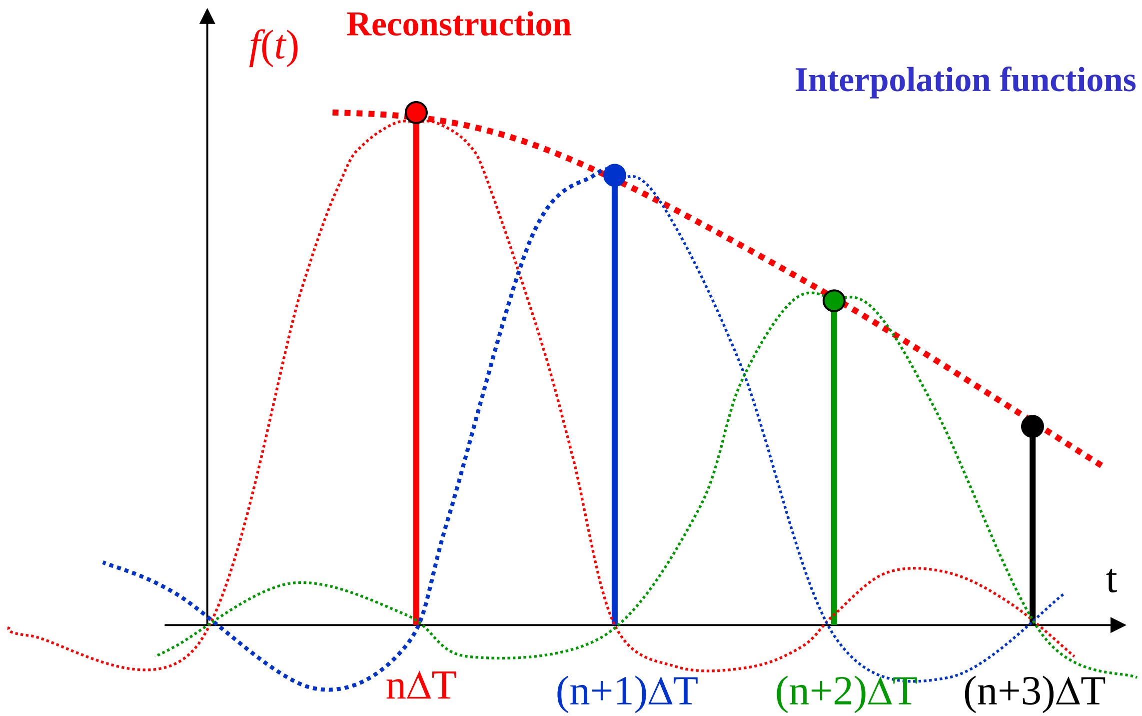
Since we cannot take infinite number of samples, approximately

$$x(t) = \sum_{k=n-l+1}^{n+l} x(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right)$$

Interpolation: viewing reconstruction in the time-domain

$$x(t) = \sum_{k=n-l+1}^{n+l} x(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right)$$

This equation suggests that original signal can be reconstructed by weighting each sample by a *sinc* function centered at the sample time and summing. This operation is illustrated in the figure below



Example 38.1

Given the signal

$$x(t) = 6\cos(10\pi t) = 6\cos(2\pi(5)t)$$

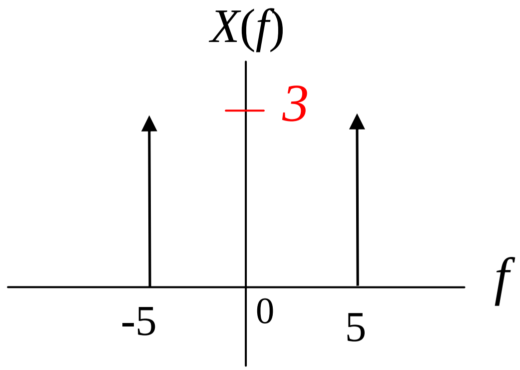
To illustrate the idea of proper reconstruction and aliasing, two different sampling frequencies are considered (f_{s1}=7 Hz, and f_{s2}=14 Hz).

The highest frequency of the signal under consideration –and the only frequency- is 5 Hz.

The objective is to see the effect of sampling a signal at both a frequency less and greater than twice the heist frequency.

By Fourier transform

$X(f) = 3\delta(f - 5) + 3\delta(f + 5)$

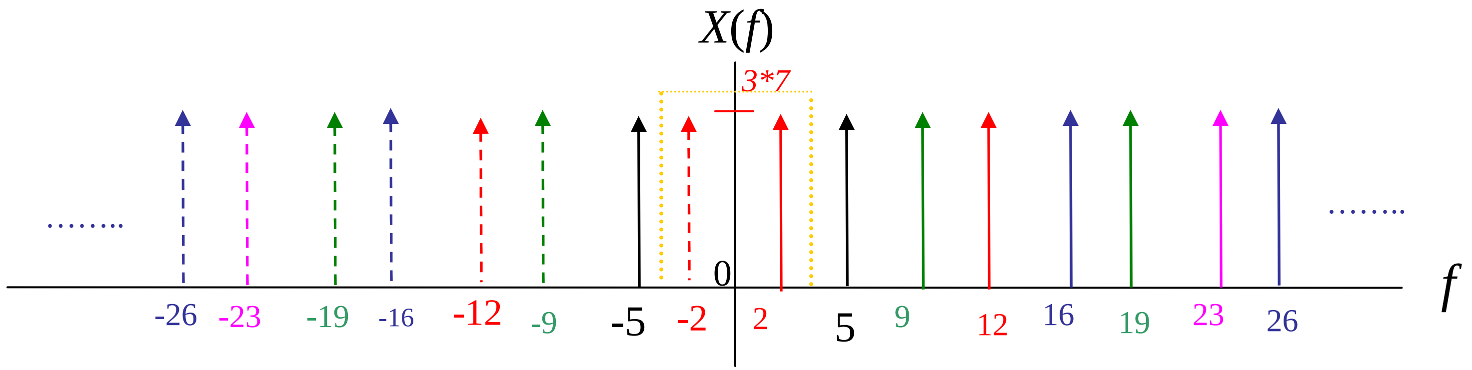


From which, the spectrum of the sampled signal can be easily found

Utilizing $f_s(t) = \Delta T \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$

$$Xs(f) = 3f_s \sum_{n=-\infty}^{\infty} \left[\delta(f - 5 - nf_s) + \delta(f + 5 - nf_s)\right]$$

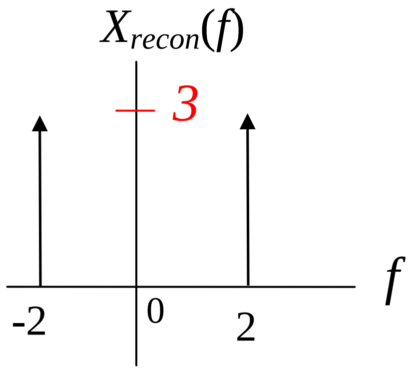
For the case of f_s= 7 Hz



A low-pass filter with cut-off frequency $=f_s/2=7/2=3.5$ Hz is used. The amplitude of the filter in the low-pass region should be $1/f_s=1/7$.

Please try to animate the filter & if possible the way the spectrum is constructed from the original signal spectrum

The reconstructed spectrum is shown



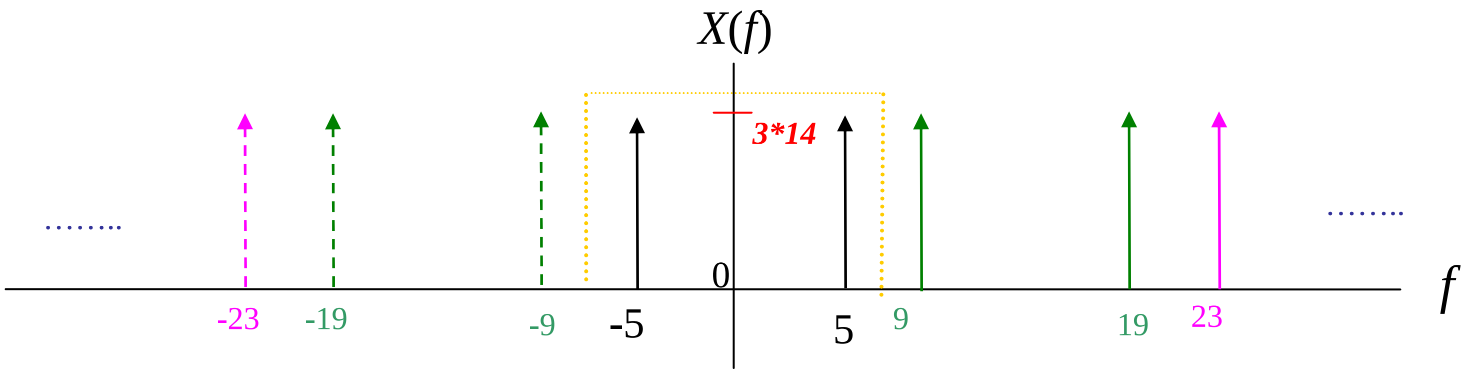
This is equivalent in the time domain to

$$x(t) = 6\cos\left(4\pi t\right) = 6\cos\left(2\pi(2)t\right)$$

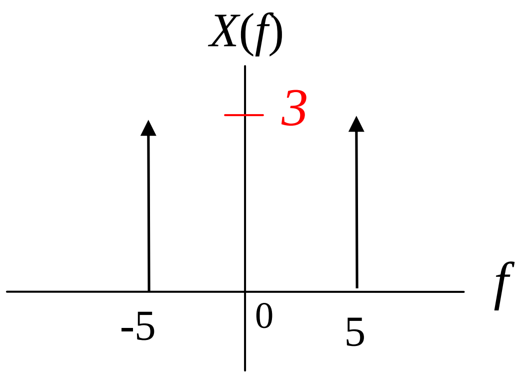
Because the original signal was sampled below Nyquist rate it could not be reconstructed properly.

Note that the reconstructed signal is similar to the original one with lower frequency as a result of aliasing.

Now, let the sampling frequency be 14Hz which above Nyquist rate. The spectrum of the sampled signal becomes



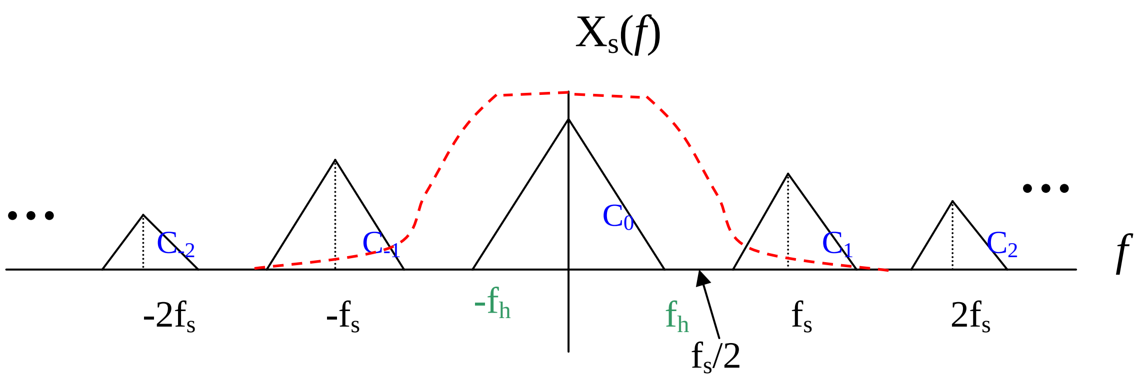
Now, a low-pass filter with cut-off frequency $=f_s/2=7/2=7$ Hz. The amplitude of the filter in the low-pass region should be $1/f_s=1/14$. The reconstructed spectrum is exactly like the original signal.



Please try to animate the filter & if possible the way the spectrum is constructed from the original signal spectrum

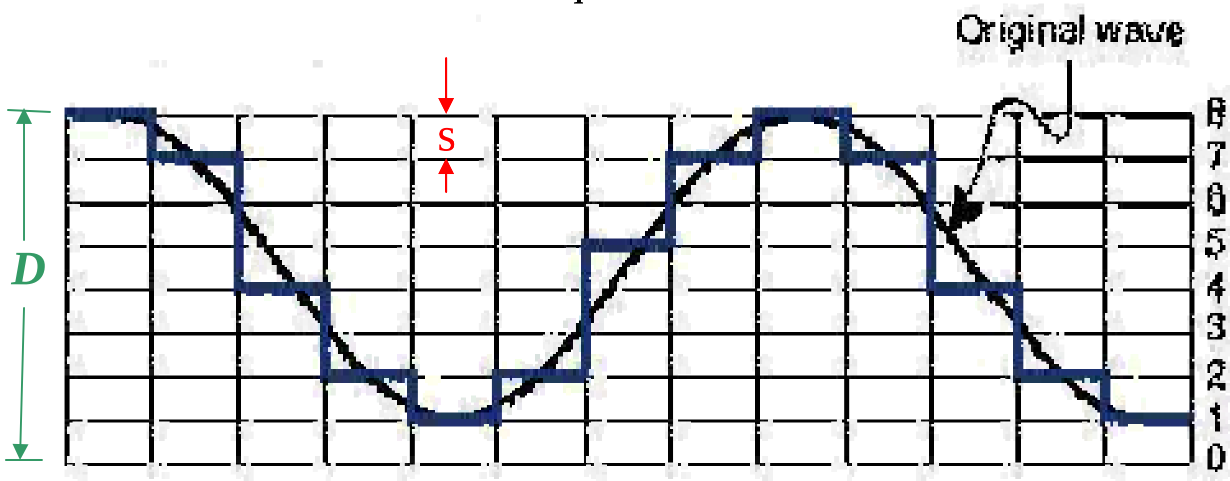
Practical reconstruction

- There are other different methods to reconstruct the signals which are not exact:
  - o In the time-domain one may use linear interpolation between the points. Other averaging techniques are also possible.
  - O In frequency-domain, RC circuit might be used to approximate low-pass filter.
    - ■ As shown in the figure below the reconstructed spectrum may suffer from variation in the amplitude in the pass-band region in addition to non-zero amplitude in the stop-band region. *Animate if possible*



Quantizing Error

- Quantization is done by rounding of the sample values to the nearest quantization level. This rounding introduces errors. In the following we look at the error introduced as a result of the quantization process.
- In general, error introduced by quantization cannot be illuminated in the reconstruction process
- The figure below illustrate the case of 8 quantization levels



n: number of bits (digital world length). In the figure *n*=3

S: width of, the quantization interval

⇨ max error induced by quantizing a sample is +1/*S* or -1/*S* *q*: the number of quantization levels. In the figure q=8.

$$q = 2^n$$

D: dynamic range of the A/D converter

$$D = \max\left[x(t)\right] - \min\left[x(t)\right]$$

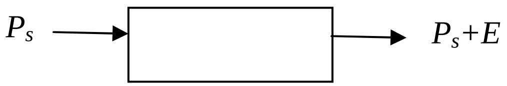
E: the mean-square error (noise power)

It is possible to show that

$E = \frac{S^2}{12}$ but $S = \frac{D}{2^n} = D\,2^{-n}$ hence we may write

$E = \frac{D^2}{12} 2^{-2n}$ (noise power)

The output is assumed to have the same P_s (signal power) in addition to sum noise



SNR: signal to noise ratio

$$SNR = \frac{P_s}{E} = \frac{P_s}{\left(D^2/12\right)2^{-2n}} = 12 P_s D^{-2} 2^{2n}$$

Usually decibels are used to represent *SNR*

$$(SNR)_{dB} = 10 \log_{10}(SNR) = 10 \log_{10}(12) + 10 \log_{10} P_s - 20 \log_{10} D + 20n \log_{10} 2$$

Which is equivalent to

$$(SNR)_{dB} = 10.79 + 6.02n + 10 \log_{10} P_s - 20 \log_{10} D$$

What is the effect of increasing n, f_s, or D?

Example

Assume the signal to be quantized is a sinusoidal signal

$$x(t) = A\cos(\omega t)$$

The signal power

$$P_s = \frac{A^2}{2}$$

Assuming we utilize the entire dynamic range of the quantizer. The dynamic range of the sinusoidal signal is the peak-to-peak value of the signal. Therfore

$$D = 2A$$

Substituting the equation for the SNR in dB

$$(SNR)_{dB} = 10.79 + 6.02n + 20\log_{10} A - 3.01 - 6.02 - 20\log_{10} A = 1.76 + 6.02n$$

Conclusions Related To Quantization Error:

From the previous example for a sinusoidal input, the SNR at the output of the quantizer is

$$(SNR)_{dB} = 1.76 + 6.02n$$

We conclude that for a sinusoidal input:

- The signal-to-noise ratio at the output of the A/D converter increases by approximately 6dB for each added bit of wordlength. This illustrates the importance of using enough number of bits.
- The effect of not using the full range of the quantizer is an effective decrease in q and thus the wordlength n. It is important that the peak-to-peak value of the signal being quantized span the full range (qS) of the quantizer.

The previous conclusions were derived for a sinusoidal input. Similar results can be deduced for other signals with minor differences.

For the rest of the discussion quantization noise are assumed negligible.

$x(t)$

Self Test:

Consider the following signal,

$$x(t) = 4\cos\left(8\pi t\right) + 6\cos\left(6\pi t\right)$$

a) What is the minimum required sampling frequency to avoid aliasing?

b) If the signal is sampled at a rate of 10 samples/second, What are the possible bandwidths of the low-pass filter required to reconstruct $x(t)$ from $x_s(t)$?

Answer:

a) Greater than twice the highest frequency=2*4=8 Hz.

b) If we sketch the spectrum of the sampled signal. It is easy to see that the bandwidth should be between 4 & 6 Hz.

The student is encouraged to sketch the spectrum of $x(t)$ and the spectrum of $x_s(t)$

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