Random Processes

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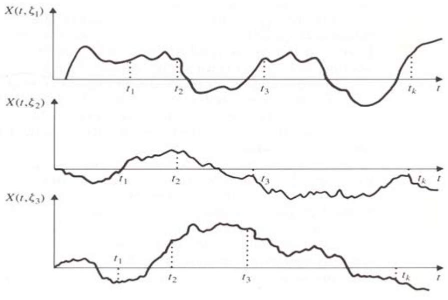
Introduction

- Real life: time (space) waveform (desired +undesired)
- Our progress and development relays on our ability to deal with such wave forms.
- The set of all the functions that are available (or the menu) is call the **ensemble** of the random process.

The graph of the function $X(t, s)$, versus t for s fixed, is called a realization, Sample path, or sample function of the random process.

For each fixed from the indexed set $I, X(t_k, s)$ is a random variable



Formal Definition

- Consider a random experiment specified by the outcomes S from some sample space S , and by the probabilities on these events.
- Suppose that to every outcome $s \in S$,we assign a function of time according to some rule: $X(t,s)$, $t \in I$.
- We have created an indexed family of random variables, $\{X(t,s), t \in I\}$.
- This family is called a random process (stochastic processes).
- We usually suppress the s and use $X(t)$ to denote a random process.
- A stochastic process is said to be discrete-time if the index set I is a countable set (i.e., the set of integers or the set of nonnegative integers). $X(nT)$ *or* $X[n]$
- A continuous-time stochastic process is one which I is continuous (thermal noise)

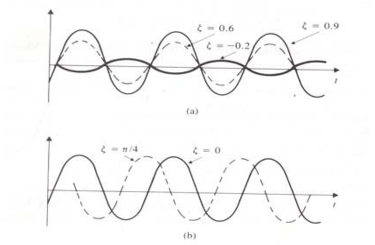
Deterministic and non-deterministic Processes

- Non-deterministic: future values **cannot** be predicted from current ones.
  - most of the random processes are non-deterministic.
- Deterministic:
- like :

$$X(t,s) = s\cos(2\pi t) \qquad -\infty < t < \infty$$

$$Y(t,s) = \cos(2\pi t + s)$$

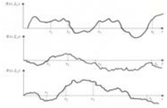
Sinusoid with random amplitude [-1,1]



Sinusoid with random phase $(-\pi, +\pi)$

Distribution and Density Functions

- A r.v. is fully characterized by a pdf or CDF. How do we characterize random processes?
- To fully define a random processes, we need N dimensional joint density function.



- Distribution and Density Functions
- **First order:**
  - $F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$
- **Second-order** joint distribution function
  - $F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$
- **Nth order** joint distribution function
- $F_X(x_1, \ldots, x_N\,; t_1, \ldots, t_N) = P\{X(t_1) \leq x_1, \ldots., X(t_N) \leq x_N\}$
- $f_X(x_1, \ldots, x_N\,; t_1, \ldots, t_N) = \frac{\partial}{\partial x_1 \ldots \partial x_N} F_X(x_1, \ldots, x_N\,; t_1, \ldots, t_N)$

Stationary and Independence

- Statistical Independence
- $f_{XY}(x_1, \ldots, x_N\,, y_1, \ldots, y_M\,; t_1, \ldots, t_N, \acute{t}_1, \ldots, \acute{t}_M)= f_X(x_1, \ldots, x_N\,,; t_1, \ldots, t_N\,) f_Y(y_1, \ldots, y_M\,;\ \acute{t}_1, \ldots, \acute{t}_M)$
- Stationary
  - If all statistical properties do not change with time
- First order Stationary Process
- $f_X(x_1; t_1) = f(x_1; t + \Delta)$, stationary to order one
- => $E[X(t)] = \overline{X}\ = constant$
- Proof
  - $X_1 = X(t_1), X_2 = X(t_2)$
  - $E[X_1] = E[X(t_1)] = \int_{-\infty}^{+\infty} x_1 f_X(x_1; t_1) dx_1$
  - $E[X_2] = E[X(t_2)] = \int_{-\infty}^{+\infty} x_2 f_X(x_2; t_2) dx_2$
  - Let $t_2 = t_1 + \Delta$
  - $E[X(t_1 + \Delta)] = E[X(t_1)]$

Cyclostationary

A discrete-time or continuous-time random process $X(t)$ is said to be cyclostationary if the joint cumulative distribution function of any set of samples is invariant with respect to shifts of the origin by *integer multiples of some period*

$$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k)$$
$$=F_{X(t_1+mT),X(t_2+mT),\ldots,X(t_k+mT)}(x_1,x_2,\ldots,x_k).$$

For all k, m and all choices of sampling times $t_1, t_2, \ldots\ t_k$

We say that $X(t)$ is wide-sense cyclostationary if the mean and autocovariance functions are invariant with respect to shifts in the time origin by integer multiples of T, that is , for every integer m.

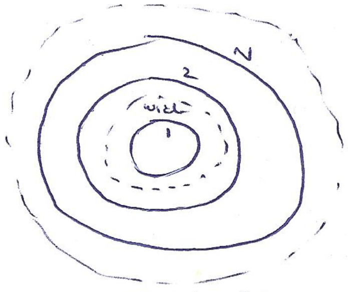
$$m_X(t+mT)=m_X(t)$$
$$C_X(t_1+mT,t_2+mT)=C_X(t_1,t_2).$$

Note that if $X(t)$ is cyclostationary, then if follows that $X(t)$ is also wide-sense cyclostationary.

N - order and Strict - Sense Stationary

Stationary to order N

$$f_X(x_1,\dots x_N ; t_1 \dots t_N) = f_X(x_1,\dots x_N ; t_1+\Delta, \dots t_N+\Delta)$$



Time Averages and Ergodicity

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}[\cdot]\,dt \quad = \bar{x} = A[x(t)]$$

time average notation similar to $E[\cdot]$

small letter

$$\mathcal{R}_{xx}(\tau) = A[x(t)\,x(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

script

when we consider all samples $\bar{x}$ & $\mathcal{R}_{xx}(\tau)$ are random Variabl.

if $E[\bar{x}] = \bar{X}$ & $E[\mathcal{R}_{xx}(\tau)] = R_{XX}(\tau)$] Ergodic Process.

small letter

Time averages equals to statistical averages.

Ergodicity is a very restritive form of stationarity.

(in real life we are usually forced to work with one sample.).

Jointly ergodic individuall ergodic + $\mathcal{R}_{xy}(\tau) = R_{XY}(\tau)$

time Cross Correlation
= Statistical Cross Correlation

Mean Ergodic Processes = ergodic in the mean.

$E[X(t)] = \bar{X} = A[x(t)] = \bar{x}$ with probability = 1

- proof in the book. (auto covariance $C_{XX}(t, t+\tau)$.

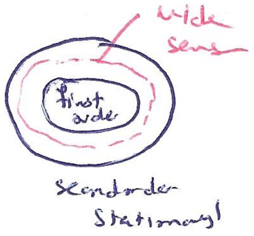
Correlation Ergodic Process $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt = R_{XX}(\tau)$

Cross correlation ergodic.

Second-Order and Wide-Sense Stationarity

- Stationary to order two

$$f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2 ; t_1+\Delta, t_2+\Delta) \quad \text{for all } t_1, t_2 \text{, and } \Delta$$



$$R_{XX}(t_1, t_2) = E[X_1 X_2] = E[X(t_1) X(t_2)]$$

if second order ~~wide se~~ stationary

$\tau = t_2 - t_1$

$$\Rightarrow R_{XX}(t_1, t_1+\tau) = E[X(t_1) X(t_1+\tau)] = R_{XX}(\tau)$$

Many practical problem requires that the mean and auto correlation be satiomany ⇒ less restrective than 2nd order stationality.

$\boxed{\text{Wide Sense Stationary}} \rightarrow$ $E[X(t)] = \bar{X} = \text{constant}$

$E[X(t)X(t+\tau)] = R_{XX}(\tau)$

Example show that the random process

$X(t) = A\cos(\omega_0 t + \Theta)$ $\longrightarrow$ A, ω_0 constants

Θ uniform $(0, 2\pi)$

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta) \overset{\text{pdf}}{\frac{1}{2\pi}} d\theta = 0$$

$$R_{XX}(t, t+\tau) = E[A\cos(\omega_0 t + \theta)\, A\cos(\omega_0 t + \omega_0 \tau + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)]$$

$$= \underset{\text{No } \underline{t}}{\frac{A^2}{2}\cos(\omega_0 \tau)} + \underbrace{\frac{A^2}{2} E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)]}_{\text{zero.}}$$

Jointly wide sense stationary X, Y

No need → ✗ ~~$R_{XY}(t_1, t_2) = E[X(t_1) Y(t_2)]$~~ $\tau = t_2 - t_1$

✓ $R_{XY}(t, t+\tau) = E[X(t) Y(t+\tau)] = R_{XY}(\tau)$

Correlation Functions

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = E[X(t)X(t+\tau)]$$

$t_1 = t$, $t_2 = t_1 + \tau$

$R_{XX}(\tau)$ ← w.s.s.

Three properties $|R_{XX}(\tau)| \leq R_{XX}(0)$

$R_{XX}(-\tau) = R_{XX}(\tau)$ ← even symmetry.

$R_{XX}(0) = E[X^2(t)]$ ← mean squared value "power"

additional properties.

(4) $E[X(t)] = \bar{X} \neq 0$ & $X(t) = \bar{X} + N(t)$

X(t) has no periodic component.

N(t) zero mean. $R_{NN}(\tau) \to 0$ as $|\tau| \to \infty$

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

(5) X(t) has periodic component.

⇒ $R_{XX}(\tau)$ " " " with same period.

(6) if X(t) is ergodic, zero mean, no periodic component.

$$\Rightarrow \lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

(7) $R_{XX}(\tau)$ cannot have arbitrary shape. (IFT of power density spectrum).

$\boxed{\text{Example}}$ for stationary ergodic process.. find mean & variance.

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

(sign ?)

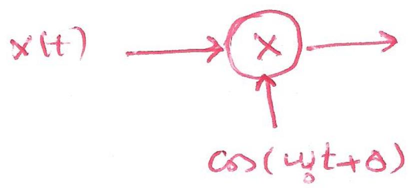
using property 4: $E[X(t)] = \bar{X} = \sqrt{25} = \pm 5$

$E[X^2(t)] = R_{XX}(0) = 25 + 4 = 29$

$\sigma_X^2 = E[X^2(t)] - (E[X(t)])^2 = 29 - 25 = 4$

Example 2: $X(t)$ W.S.S.

$$R_{XX}(\tau) = e^{-a|\tau|} \quad a>0 \text{ constant.}$$



$$Y(t) = X(t)\cos(\underset{\uparrow\ \text{constan}}{w_0}t+\underset{\uparrow}{\theta})$$

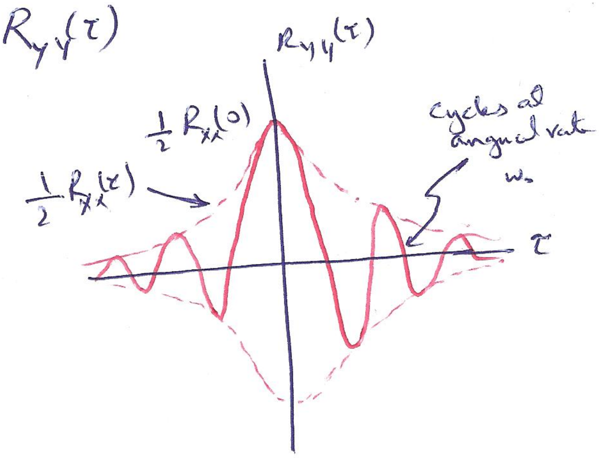
$U(-\pi,\pi)$ independent of $X(t)$

$$R_{YY}(t,t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[X(t)\,X(t+\tau)\cos(w_0t+\theta)\cos(w_0t+w_0\tau+\theta)]$$

$$= R_{XX}(\tau)\,\frac{1}{2}\,E[\cos(w_0\tau)+\underbrace{\cos(2w_0t+w_0\tau+2\theta)}_{\underline{E}\ \text{zero}}]$$

$$R_{YY}(t,t+\tau) = \frac{1}{2}R_{XX}(\tau)\cos(w_0\tau) = R_{YY}(\tau)$$

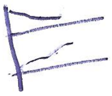


Cross-Correlation Function and its Properties.

$$E[X(t)Y(t+\tau)]$$

$$= R_{XY}(t,t+\tau) = 0 \quad \text{if orthogonal process.}$$

$$\text{if} \qquad = E[X(t)]\,E[Y(t+\tau)] \quad \text{independent.}$$



if independent + Wide S.S.

$$R_{XY}(\tau) = \bar{X}\,\bar{Y} \quad \text{Constant.}$$

Some properties for R_{XY} (W.SS).

(1) $R_{XY}(-\tau) = R_{YX}(\tau)$

(2) $|R_{XY}(\tau)| \leqslant \sqrt{R_{XX}(0)R_{YY}(0)}$ ← geometric mean (tighter).

(3) $|R_{XY}(\tau)| \leqslant \frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$ ← arithmetic Mean

bounds.

$$\sqrt{R_{XX}(0)+R_{YY}(0)} \leqslant \frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$$

Example $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

(Many present) $Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$

ω_0 Constant. A & B r.Ds.

if A & B are uncorrelated, zero-mean. with same variance $X(t)$ is w.s.s.

" " " " " $Y(t)$ " "

show that $X(t)$ & $Y(t)$ are jointly w.s.s.

$$R_{xy}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

substitute $X(t)$ & $Y(t)$

sin differ

$$= E[AB]\cos(2\omega_0 t + \omega_0\tau)$$

$0 \rightarrow$

$$+E[B^2]\sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)$$

$\sigma^2 \rightarrow$

$$-E[A^2]\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau)$$

$\sigma^2 \rightarrow$

$$= -\sigma^2 \sin(\omega_0\tau)$$

← only depends on τ

← not even; max not at zero (cross correlation)

Covariance Matrix

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - \overset{\text{mean}}{E[X(t)]}\}\{X(t+\tau) - \overset{\text{mean}}{E[X(t)]}\}\right]$$

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]E[X(t+\tau)]$$

we can define cross covariance $C_{XY}(t, t+\tau)$ ← in the same way

for joint wide sense stationary

$$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$$

Variance = Covariance at $\tau = 0$

for w.s.s. not function of time.

$$\sigma_X^2 = R_{XX}(0) - \bar{X}^2$$

if $C_{XY}(t, t+\tau) = 0$ uncorrelated $\Rightarrow R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)]$

independent $\Rightarrow$ uncorrelated

$\Leftarrow$

not true

only true for Jointly Gaussian

8/6

Discrete Time Processes & Sequences.

Same applies replace τ with kT_s

t with nT_s

or omitt T_s

$$\text{Mean} = E[X(nT_s)]$$

$$R_{XX}(nT_s, nT_s + kT_s) = E[X(nT_s)\,X(nT_s + kT_s)]$$

$$\vdots$$

Cross-Correlation Function and its properties

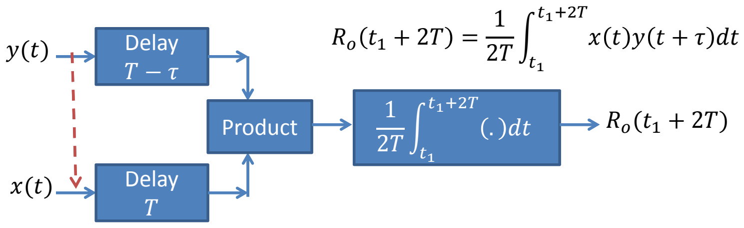
- $R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$
- If X and Y are jointly w.s.s. we may write $R_{XY}(\tau)$.
- Orthogonal processes $R_{XY}(t, t+\tau)$ =0
- If X and Y are statistically independent
  - $E[X(t)Y(t+\tau)]$=$E[X(t)]E[Y(t+\tau)]$
- If in addition to being independent they are at least w.s.s.
  - $E[X(t)Y(t+\tau)] = \bar{X}\bar{Y}$

Some properties for R_{XY}

- $R_{XY}(-\tau) = R_{YX}(\tau)$
- $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$
- $|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$
- The geometric mean is tighter than the arithmetic mean
- $\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

Measurement of Correlation Function

- In real life, we can never measure the true correlation .
- We assume ergodicity and use portion of the available time.
- Assume ergodicity, no need to prove mathematically "physical sense"
- Assume jointly ergodic => stationary
- Let $t_1 = 0$, $R_0(2T) = R_{xy}(\tau) = R_{XY}(\tau)$
- Similarly, we may find $R_{XX}(\tau)$ & $R_{YY}(\tau)$



$$R_o(t_1 + 2T) = \frac{1}{2T}\int_{t_1}^{t_1+2T} x(t)y(t+\tau)dt$$

Example

- Use the above system to measure the $R_{XX}(\tau)$ for $X(t) = Acos(\omega_0 + \theta)$.
- $R_o(2T) = \frac{1}{2T}\int_{-T}^{T} A^2 \cos(\omega_0 t + \theta)\cos(\omega_0 t + \theta + \omega_0 \tau)\, dt$
- $= \frac{A^2}{2T}\int_{-T}^{T}[\cos(\omega_0 \tau) + \cos(2\omega_0 t + 2\theta + \omega_0 \tau)]dt$
- $= R_{XX}(\tau) + \epsilon(T)$ where $R_{XX}(\tau) = \frac{\mathrm{A}^2}{2}\cos(\omega_0 \tau)$
- $\epsilon(T) = \frac{\mathrm{A}^2}{2}\cos(\omega_0 \tau + 2\theta)\frac{\sin 2\omega_0 T}{2\omega_0 T}$
- If we require the $\epsilon(T)$ to be at least 20 times less than the largest value of the true autocorrelation $|\epsilon(T)| < 0.05 R_{XX}(0)$
- $\frac{1}{2\omega_0 T} \leq 0.05 \Rightarrow T \geq \frac{10}{\omega_0}$
- Wait enough time! Depending on the frequency

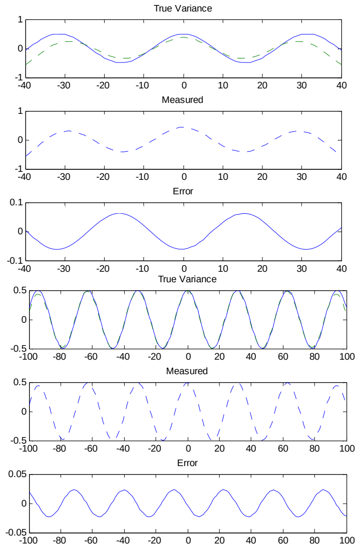
Matlab: Measuring the correlation

```
% Dr. Ali Muqaibel
% Measurement of Correlation function
clear all
close all
clc
T=100;
A=1;
omeg=0.2;
t=-T:T;
thet=2*pi*rand(1,1);
X=A*cos(omeg*t+thet);
[R,tau]=xcorr(X,'unbiased');
%R=R/(2*T);
True_R=A^2/2*cos(omeg*tau);
Err=A^2/2*cos(omeg*tau+2*thet)*sin(2*omeg*T)/(2*omeg*T);
subplot(3,1,1)
plot(tau,True_R,tau,R+Err,':')
title ('True Variance')
subplot(3,1,2)

plot (tau,R,':')
title ('Measured')
% error

subplot (3,1,3)
plot (tau,Err)
title ('Error')
% error
```

T=20, OMEGA=0.2



Note the error is less than 5%

T=50, OMEGA=0.2

Gaussian Random Processes

A random process $X(t)$ is a Gaussian random process if the samples

$$X_1 = X(t_1), X_2 = X(t_2), \dots X_t = X(t_k)$$

are **jointly Gaussian** random variables for all k ,and all choices of $t_1, t_2, \dots t_k$.
This definition applies for discrete-time and continuous-time processes.
The joint pdf of jointly Gaussian random variables is determined by the vector of means and by the covariance matrix:

$$f_{X_1,X_2,\dots\dots,X_k}(x_1,\dots\dots,x_k) = \frac{e^{-\frac{1}{2}(X-m)^T C^{-1}(X-m)}}{(2\pi)^{k/2}|C|^{\frac{1}{2}}}$$

where

$$\mathrm{m} = \begin{bmatrix} m_X(t_1) \\ . \\ . \\ . \\ m_X(t_k) \end{bmatrix} \qquad C = \begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & . & . & . & C_X(t_{1,}t_k) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & . & . & . & C_X(t_2,t_k) \\ . & . & & & & . \\ . & . & & & & . \\ . & & & & & \\ C_X(t_{k,}t_1) & \dots & & & & C_X(t_{k,}t_k) \end{bmatrix}$$

Example iid Gaussian Sequence

Let the discrete-time random process X_n be a sequence of independent Gaussian random variables with mean m and variance σ^2
The covariance matrix for the times $t_1, \ldots, t_k$ is

$$\{C_X(t_1, t_j)\} = \{\sigma^2 \delta_{ij}\} = \sigma^2 I,$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise, and I is the identity matrix. Thus the corresponding joint pdf is

$$f_{X_1,\ldots\ldots X_k}(x_1, x_2, \ldots\ldots, x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\left\{-\sum_{i=1}^{k}(x_i - m)^2 / 2\sigma^2\right\}$$

$$= f_X(x_1) f_X(x_2) \ldots f_X(x_k)$$

Example of a Gaussian Random Process

- A Gaussian Random Process which is W.S.S. $\bar{X} = 4$ and $R_{XX}(\tau) = 25e^{-3|\tau|} + 16$
- Specify the joint density function for three r.v. $X(t_i), i = 1,2,3 \ldots, t_i = t_0 + \left[\frac{i-1}{2}\right], t_0 is\ constant$
- $t_k - t_i = \frac{k-i}{2}, i\ and\ k = 1,2,3, \ldots$
- $R_{XX}(t_k - t_i) = 25e^{-\frac{3|k-i|}{2}} + 16$
- $C_{XX}(t_k - t_i) = 25e^{-\frac{3|k-i|}{2}} + 16 - (4)^2$
- $[C_X] = 25\begin{bmatrix} 1 & e^{-\frac{3}{2}} & e^{-\frac{6}{2}} \\ e^{-\frac{3}{2}} & 1 & e^{-\frac{3}{2}} \\ e^{-\frac{6}{2}} & e^{-\frac{3}{2}} & 1 \end{bmatrix}$

Complex Random Processes

- A complex random process $Z(t)$is given by
- $Z(t) = X(t) + jY(t)$
- $R_{ZZ}(t, t+\tau) = E[Z^*(t)Z(t+\tau)]$
- $C_{ZZ}(t, t+\tau) =$ $E[\{Z(t) - E[Z(t)]\}^*\{Z(t+\tau) - E[Z(t+\tau)]\}]$
- Note the conjugate
- There could be a factor of $\frac{1}{2}$ in some books
- See example in Peebles

Example Signal Plus Noise

Suppose we observe a process $Y(t)$, which consists of a desired signal $X(t)$ plus noise $N(t)$.

Find the cross-correlation between the observed signal and the desired signal assuming that $X(t)$ and $N(t)$ are independent random processes.

$$
\begin{aligned}
R_{X,Y}(t_1,t_2) &= E[X(t_1)Y(t_2)] \\
&= E[X(t_1)\{X(t_2)+N(t_2)\}] \\
&= E[X(t_1)X(t_2)] + E[X(t_1)N(t_2)] \\
&= R_{XX}(t_1,t_2) + E[X(t_1)]E[N(t_2)] \\
&= R_{XX}(t_1,t_2) + m_X(t_1)m_N(t_2)
\end{aligned}
$$

where the third equality followed from the fact that $N(t)$ and $X(t)$ are independent.

EXAMPLES OF DISCRETE_TIME & Continuous-Time RANDOM PROCESSES

See *Leon Garcia*
Probability, Statistics, and Random Processes
for Electrical Engineers, 3rd Edition

9.5 GAUSSIAN RANDOM PROCESSES,WIENER PROCESS,AND BROWNIAN MOTION

EXAMPLES OF DISCRETE_TIME RANDOM PROCESSES

iid Random Processes

Let X_n be a discrete-time random process consisting of a sequence of independent, identically distributed (iid) random variables with common cdf $F_X(x)$ mean m and variance σ^2. The sequence X_n is called the iid random process. The joint cdf for any time instants $n_1, \dots, n_k$ is given by

$$F_{X_1,\dots X_K}(x_1, x_2, \dots, x_k) = P[X_1 \le x_1, X_2 \le x_2, \dots, X_k \le x_k)$$
$$= F_X(x_1)F_X(x_2)\dots F_X(x_k),$$

where for simplicity X_k denotes X_{n_k}. The equation above implies that if X_n is discrete-values, the joint pmf factors into the product of individual pmf's, and if X_n is continuous-valued, the joint pdf factors into the product of the individual pdf's.

The *mean of an iid process* is obtained

$$m_X(n) = E[X_n] = m \qquad \text{for all } n$$

Thus, the mean is constant.

The autocovariance function is obtained from as follows. If $n_1 \neq n_2$, then

$$C_X(n_1,n_2) = E[(X_{n_1}-m)(X_{n_2}-m)]$$
$$= E[(X_{n_1}-m)]E[(X_{n_2}-m)] = 0$$

since X_{n_1} and X_{n_2} are independent random variables. If $n_1 = n_2 = n$ then

$$C_X(n_1,n_2) = E[(X_n-m)^2] = \sigma^2$$

We can express the autocovariance of the iid process in compact form as follows:

$$C_X(n_1,n_2) = \sigma^2 \delta_{n_1 n_2},$$

where $\delta_{n_1 n_2} = 1$ if $n_1 = n_2$ and 0 otherwise

The autocorrelation function of the iid process is:

$$R_X(n_1,n_2) = C_X(n_1,n_2) + m^2$$

Example : Bernoulli Random Process

Let I_n be a sequence of independent Bernoulli random variables. I_n is then an iid random process taking on values from the set {0,1}. A realization of such a process is shown in Figure.

For example, I_n could be an indicator function for the event " a light bulb fails and is replaced on day *n*."

Since I_n is a Bernoulli random variable, it has mean and variance

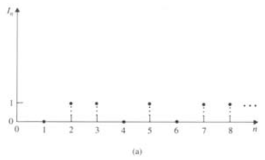
$$m_I(n) = p$$
$$VAR[I_n] = p(1-p)$$

The independence of the I_n makes probabilities easy to compute. For example, the probability that the first 4 bits in the sequence are 1001 is

$$P[I_1 = 1, I_2 = 0, I_3 = 0, I_4 = 1] = P[I_1 = 1]P[I_2 = 0]P[I_3 = 0]P[I_4 = 1]$$
$$= p^2(1-p)^2$$

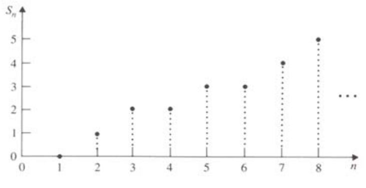
Similarly, the probability that the second bit is 0 and the seventh is 1 is

$$P[I_2 = 0, I_7 = 1] = P[I_2 = 0]P[I_7 = 1] = p(1-p)$$



Realization of a Bernoulli process. $I_n = 1$ indicates that a light bulb fails and is replaced in day n.

Sum Processes: The Binomial Counting and Random Walk Processes

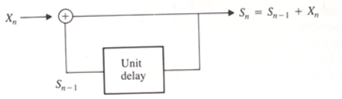


(b)

(b) Realization of a binomial process. S_n denotes the number of light bulbs that have failed up to time *n.*

Many interesting random processes are obtained as the sum of a sequence of iid random variables, $X_1, X_2, \ldots$.

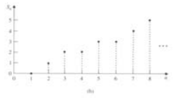
$$S_n = X_1 + X_2 + \cdots . X_n = S_{n-1} + X_n, \qquad n = 1,2, \ldots$$



The sum process $S_n = X_1 + \cdots + X_n$ $S_0 = 0$, can be generated in this way.

where $S_0 = 0$. We call S_n the sum process. The pdf or pmf of S_n is found using the convolution .
Note that S_n depends on the "past," $S_1, \ldots, S_{n-1}$ only through S_{n-1} , that is, S_n is independent of the past when S_{n-1} is known. This can be seen clearly from the previous Figure, which shows a recursive procedure for computing S_n . Thus S_n is a **Markov process**.

Example Binomial Counting Process



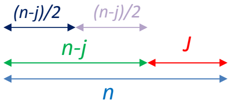
Let the I_i be the sequence of independent Bernoulli random variables in a previous Example, and let $S_n$be the corresponding sum process. S_n is then the ***counting process*** that gives the number of successes in the first n Bernoulli trials. The sample function for S_n corresponding to a particular sequence of $I_i's$ is shown in the Figure up. If I_n indicates that a light bulb fails and is replaced on day *n*, then S_n denotes the number of light bulbs that have failed up to day $n.$
Since S_n is the sum of n independent Bernoulli random variables, S_n is a binomial random variable with parameters *n* and $p = P[I = 1]$

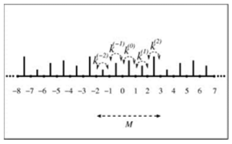
$$P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j} \qquad \text{for } 0 \le j \le n,$$

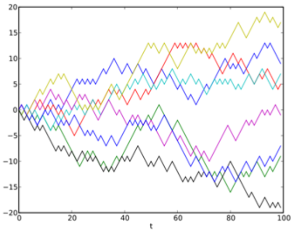
and zero otherwise. Thus S_n has mean np and variance $np(1-p)$.
Note that the mean and variance of this process grow linearly with time (n).

Example One-Dimensional Random Walk

Let D_n be the iid process of ± 1 random variable as in the previous example, and let S_n be the corresponding sum process. S_n is then the position of the particle at time *n.*
The random process S_n is an example of a one-dimensional random walk.
A sample function of S_n is shown in the Figure
The pmf of S_n is found as follows. If there are k $"+1"$ in the first n trials, then there are $n-k$ $"-1"$ and $S_n = k-(n-k) = 2k-n$.
Conversely, $S_n = j$ if the number of "$+1$"s is $k = j + \frac{n-j}{2} = \frac{\boldsymbol{j+n}}{\boldsymbol{2}}$.
If $\frac{j+n}{2}$ is not an integer, then S_n cannot equal j.
Thus $P[S_n = 2k-n] = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0,1,\dots,n$







Example Sum of iid Gaussian Sequence

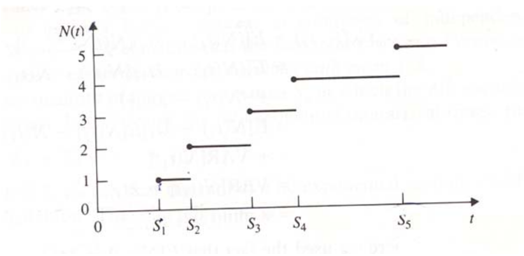
Let X_n be a sequence of iid Gaussian random variables with zero mean and variance σ^2 . Find the joint pdf of the corresponding sum process at times n_1 and n_2.
The sum process S_n is also a Gaussian random process with mean zero and variance $n\sigma^2$. The joint pdf of $S_n$at times n_1 and n_2 is given by $n\sigma^2$

$$f_{S_{n1},S_{n2}}(y_1,y_2) = f_{S_{n2-n1}}(y_2-y_1)\, f_{S_{n1}}(y_1)$$
$$= \frac{1}{\sqrt{2\pi(n_2-n_1)\sigma^2}} e^{-(y_2-y_1)^{2/[2(n_2-n_1)\sigma^2]}} \frac{1}{\sqrt{2\pi n_1\sigma^2}} e^{-y_1^2/2n_1\sigma^2}$$

EXAMPLES OF CONTINUOUS-TIME RANDOM PROCESSES

Poisson Process

Consider a situation in which events occur at random instants of time at an average rate of a customer to a service station or the breakdown of a component in some system. Let $N(t)$ be the number of event occurrences in the time interval [0,t]. $N(t)$ is then a nondecreasing, integer-valued, continuous-time random process as shown in Figure.



A sample path of the Poisson counting process. The event occurrence times are denoted by $S_1, S_2, \ldots\ldots$The *j* th interevent time is denoted by $X_j = S_j - S_{j-1}$

Poisson Process.. From Binomial

If the probability of an event occurrence in each subinterval is *p,* then the expected number of event occurrences in the interval [0,*t*] is ***np****.* Since events occur at a rate of λ events per second, the average number of events in the interval [0,*t*] is also $\boldsymbol{\lambda t}$.Thus we must have that

$$\lambda t = np$$

If we now let $n \to \infty (i.e., \delta \to 0)$ and $p \to 0$ while $np = \lambda t$ remains fixed, then the binomial distribution approaches a Poisson distribution with parameter λt. We therefore conclude that *the number of event occurrences* $N(t)$ *in the interval* $[0, t]$ *has a Poisson distribution with mean* λt*:*

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad for\ k = 0,1, \ldots$$

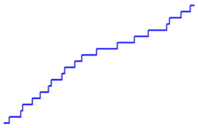
For this reason $N(t)$ is called the Poisson process.

Replace p with $\lambda\, t/n$

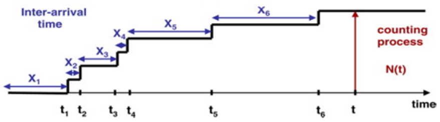
$$P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j} \qquad \text{for } 0 \le j \le n,$$

For detailed derivation , please see http://www.vosesoftware.com/ModelRiskHelp/index.htm#Probability_theory_and_statistics/Stochastic_processes/Deriving_the_Poisson_distribution_from_the_Binomial.htm

Poisson Random Process



- Also known as *Poisson Counting Process*
- *Arrival of customers, failure of parts, lightning,….internet   t>0*
- *Two conditions:*
    - *Events do not coincide.*
    - *# of occurrence in any given time interval is independent of the number in any non overlapping time interval. (independent increments)*
- *Average rate of occurrence=*λ.
- $P[X(t)=k]=\frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k=0,1,2,\ldots \quad (0,t)$
- $f_X(x)=\sum_{k=0}^{\infty}\frac{(\lambda t)^k e^{-\lambda t}}{k!}\delta(x-k)$
- $E[X(t)]=\lambda t$
- $variance=\lambda t=mean$
- $E[X^2(t)]=\lambda t[1+\lambda t]$



- The probability distribution of the waiting time until the next occurrence is an exponential distribution.
- The occurrences are distributed uniformly on any interval of time.

http://en.wikipedia.org/wiki/Poisson_process

Joint probability density function for Poisson Random Process

- The joint probability density function for the poison process at times $0<t_1<t_2$
- $P[X(t_1)=k_1]=\frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!}, \quad k_1=0,1,2,\ldots$
- The probability of k_2 occurrence over $(0,t_2)$ given that k_1 events occurred over $(0,t_1)$, is just the probability that k_2-k_1 events occurred over (t_1,t_2), which is
- $P[X(t_2)=k_2|X(t_1)=k_1]=\frac{[\lambda(t_2-t_1)]^{k_2-k_1} e^{-\lambda(t_2-t_1)}}{(k_2-k_1)!}$
- For $k_2>k_1$, the joint probability is given by
- $P(k_1,k_2)=P[X(t_2)|X(t_1)=k_1]P[X(t_1)=k_1]$
- $=\frac{(\lambda t_1)^{k_1}[\lambda(t_2-t_1)]^{k_2-k_1} e^{-\lambda t_2}}{k_1!\,(k_2-k_1)!}$
- The joint density becomes
- $f_X(x_1,x_2)=\sum_{k_1=0}^{\infty}\sum_{k_2=k_1}^{\infty}P(k_1,k_2)\delta(x_1-k_1)\delta(x_2-k_2)$
- Example : demonstrate the higher-dimensional pdf

Example I

Inquiries arrive at a recorded message device according to a Poisson process of rate 15 inquiries per minute. Find the probability that in a 1-minute period, 3 inquiries arrive during the first 10 seconds and 2 inquiries arrive during the last 15 seconds.

The arrival rate in seconds is $\lambda = \frac{15}{60} = \frac{1}{4}$ inquiries per second.
Writing time in seconds, the probability of interest is

$$P[N(10) = 3 \text{ and } N(60) - N(45) = 2]$$

By applying first the independent **increments property**, and then the **stationary increments** property, we obtain

$$\begin{aligned} P[N(10) = 3 \text{ and } N(60) - N(45) = 2] \\ &= P[N(10) = 3]P[N(60) - N(45) = 2] \\ &= P[N(10) = 3]P[N(60-45) = 2] \\ &= \frac{(10/4)^3 e^{-10/4}}{3!} \frac{(15/4)^2 e^{-15/4}}{2!} \end{aligned}$$

Example II

Find the mean and variance of the time until the arrival of the tenth inquiry in the previous Example. The arrival rate is $\lambda = 1/4$ inquiries per second, so the inter-arrival times are exponential random variables with parameter λ.

From Tables, the mean and variance of an inter-arrival time are then $1/\lambda$ and $1/\lambda^2$, respectively.

The time of the tenth arrival is the sum of ten such iid random variables, thus

$$E[S_{10}] = 10E[T] = \frac{10}{\lambda} = 40 \sec$$

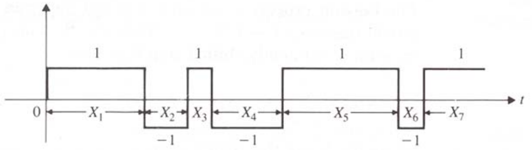
$$\text{VAR}[S_{10}] = 10\text{VAR}[\text{T}] = \frac{10}{\lambda^2} = 160 \sec^2$$

Example Random Telegraph Signal

Consider a random process $X(t)$ that assumes the values ± 1 . Suppose that $X(0) = \pm 1$ with probability $\frac{1}{2}$ and suppose that $X(t)$ then changes polarity with each occurrence of an event in a Poisson process of rate α. The next figure shows a sample function of $X(t)$.

The pmf of $X(t)$ is given by

$$P[X(t)=\pm 1]=P[X(t)=\pm 1 | X(0)=1]P[X(0)=1]$$
$$+P[X(t)=\pm 1 | X(0)=-1]P[X(0)=-1].$$



Sample path of a random telegraph signal. The times between transitions X_j are iid exponential random variables.

The conditional pmf's are found by noting that $X(t)$ will have the same polarity as $X(0)$ only when an even number of events occur in the interval $(0, t]$. Thus

$$P[X(t)=\pm 1 | X(0)=\pm 1]=P[N(t)=\text{even integer}]$$
$$=\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j}}{(2j)!}e^{-\alpha t}$$
$$=e^{-\alpha t}\frac{1}{2}\{e^{\alpha t}+e^{-\alpha t}\}$$
$$=\frac{1}{2}\{1+e^{-2\alpha t}\}$$

$X(t)$and X(0) will differ in sign if the number of events in t is odd:

$$P[X(t)=\pm 1 | X(0)=\mp 1]=\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j+1}}{(2j+1)!}e^{-\alpha t}$$
$$=e^{-\alpha t}\frac{1}{2}\{e^{\alpha t}-e^{-\alpha t}\}$$
$$=\frac{1}{2}\{1-e^{-2\alpha t}\}.$$

Mean & Variance of the Random Telegraph Signal

We obtain the pmf for $X(t)$ by substituting into :

$$P[X(t)=\pm 1] = P[X(t)=\pm 1 \mid X(0)=1]P[X(0)=1]$$
$$+ P[X(t)=\pm 1 \mid X(0)=-1]P[X(0)=-1].$$

$$P[X(t)=1] = \frac{1}{2}\frac{1}{2}\{1+e^{-2\alpha t}\} + \frac{1}{2}\frac{1}{2}\{1-e^{-2\alpha t}\} = \frac{1}{2}$$

$$P[X(t)=-1] = 1 - P[X(t)=1] = \frac{1}{2}$$

Thus the random telegraph signal is equally likely to be ± 1 at any time
The mean and variance of $X(t)$ are

$$m_X(t) = 1P[X(t)=1] + (-1)P[X(t)=-1] = 0 \qquad t>0$$

$$\mathrm{VAR}[X(t)] = E[X(t)^2] = (1)^2 P[X(t)=1]$$
$$+ (-1)^2 P[X(t)=-1] = 1$$

Auto-covariance of the Random Telegraph Signal

The autocovariance of $X(t)$ is found as follows:

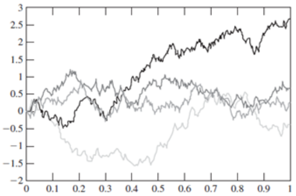
$$\begin{aligned}
C_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\
&= 1P[X(t_1)=X(t_2)] + (-1)P[X(t_1)\neq X(t_2)] \\
&= \frac{1}{2}\{1+e^{-2\alpha|t_2-t_1|}\} - \frac{1}{2}\{1-e^{-2\alpha|t_2-t_1|}\} \\
&= e^{-2\alpha|t_2-t_1|}
\end{aligned}$$

Thus time samples of $X(t)$ become less and less correlated as the time between them increases.

Wiener Process and Brownian Motion

- A continuous-time Gaussian random process as a limit of a discrete time process.
- Suppose that the <u>symmetric</u> random walk process (i.e., $p = 0.5$) takes steps of magnitude $\pm h$ every δ seconds.
- We obtain a continuous-time process by letting $X_\delta(t)$ be the accumulated sum of the random step process up to time t.
- $X_\delta(t)$ is a staircase function of time that takes jumps of $\pm h$ every δ seconds.
- At time t, the process will have taken $n = [\frac{t}{\delta}]$ jumps, so it is equal to

$$X_\delta(t) = h(D_1 + D_2 + \cdots + D_{[t/\delta]}) = hS_n.$$



- The mean and variance of $X_\delta(t) are$
  - $\text{E}[X_\delta(t)] = hE[S_n] = 0$
  - $\text{VAR}[X_\delta(t)] = h^2 n\ \text{VAR}[Dn] = h^2 n$
- We used the fact that $VAR[D_n] = 4p(1-p) = 1\ since\ p = \frac{1}{2}$
- By shrinking the time between jumps and letting $\delta \rightarrow 0$ and $h \rightarrow 0$ with $h = \sqrt{\alpha\delta}$
- $X(t)$ then has a mean and variance
  - $E[X(t)] = 0$
  - $VAR[X(t)] = \left(\sqrt{\alpha\delta}\right)^2 \left(\frac{t}{\delta}\right) = \alpha\delta$
- $X(t)$ is called the **Wiener random process**. It is used to model *Brownian motion*, the motion of particles suspended in a fluid that move under the rapid and random impact of neighboring particles.

Wiener Process

- As $X(t)$ approaches the sum of infinite number of random variables since $n = \left[\frac{t}{\delta}\right] \to \infty$
- $X(t) = \lim_{\delta \to 0} h\, S_n = \lim_{n \to \infty} \sqrt{\alpha t}\, \frac{S_n}{\sqrt{n}}$
- By the central limit theorem the pdf $X(t)$ therefor approaches that o a Gaussian variable with mean zero and variance αt :
- $f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{\frac{-x^2}{2\alpha t}}$
- $X(t)$ inherits the property of independent and stationary increments from the random walk process from which it is derived.
- The independent increments property and the same sequence of steps can be used to show that the *autocovariance of* $X(t)$ is given by
- $C_X(t_1, t_2) = \alpha \min(t_1, t_2) = \alpha t_1 \quad for\ t_1 < t_2$
- Wiener and Poisson process have the same covariance despite the fact that they are different.

Practice Problem :Poisson Process

- Suppose that a secretary receives calls that arrive according to a Poisson process with a rate of 10 calls per hour.
- What is the probability that no calls go unanswered if the secretary is a way from the office for the first and last 15 minutes of an hour?

In class practice: Wide-Sense Stationary Random Process

- Let X_n be an iid sequence of Gaussian random variables with zero mean and variance σ^2, and let Y_n be the average of two consecutive values of X_n ,

$$Y_n = \frac{X_n + X_{n-1}}{2}$$

- Find the mean of Y_n.
- Find the covariance $C_Y(i,j)$
- What is the distribution of the random variable Y_n. Is it stationary?