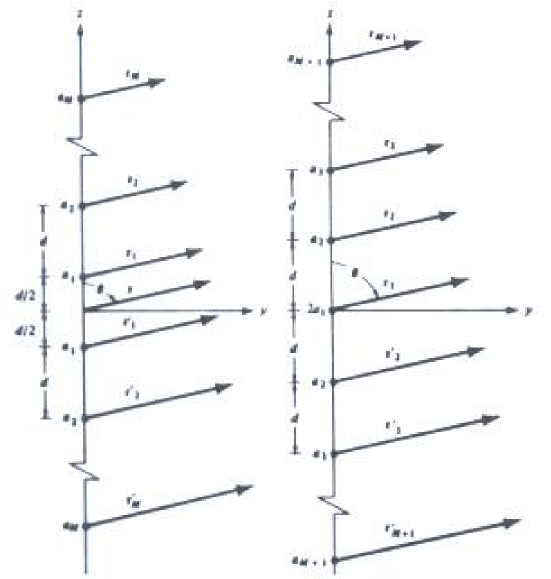
Array factor of N-element Linear Array with non-uniform amplitudes:



(a) Even number of elements (b) Odd number of elements

Figure 6.17 Nonuniform amplitude arrays of even and odd number of elements.

$$AF_{2M} = a_1 e^{j\frac{1}{2}kd\cos\theta} + a_2 e^{j\frac{3}{2}kd\cos\theta} + \cdots + a_M e^{j\frac{2M-1}{2}kd\cos\theta}$$
$$+ a_1 e^{-j\frac{1}{2}kd\cos\theta} \cdots + a_M e^{-j\frac{2M-1}{2}kd\cos\theta}$$

$$= 2\sum_{n=1}^{M} a_n \cos\left[\frac{2n-1}{2} kd\cos\theta\right]$$

Normalized $(AF_{2M})_n = \sum_{n=1}^{M} a_n \cos\left[\frac{2n-1}{2} kd\cos\theta\right] = \sum_{n=1}^{M} a_n \cos\left[(2n-1)u\right]$

where $u = \frac{1}{2} kd\cos\theta = \frac{\pi d}{\lambda}\cos\theta$

$$AF_{2M+1} = 2a_1 + a_2 e^{jkd\cos\theta} + \cdots a_{M+1} e^{jMkd\cos\theta} + a_2 e^{-jkd\cos\theta} + \cdots + a_{M+1} e^{-jMkd\cos\theta}$$
$$= 2\sum_{n=1}^{M+1} a_n \cos\left[(n-1)kd\cos\theta\right]$$

$$\therefore (AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos[2(n-1)u]$$

Binomial Array

using the coefficients of the expansion of the function $(1+x)^{m-1} = 1 + (m-1)x + \frac{(m-1)(m-2)}{2!}x^2 + \cdots$

$\therefore$ coefficients are obtained from Pascal triangle

e.g.

$m=1$	1
$m=2$	1 1
$m=3$	1 2 1
$m=4$	1 3 3 1
$m=5$	1 4 6 4 1

etc.

$\therefore$ for Two-elements $2M=2 \rightarrow a_1 = 1$

Three-elements $2M+1=3$

$\therefore 2a_1 = 2 \rightarrow a_1 = 1$

$a_2 = 1$

Four elements $2M=4$

$a_1 = 3, \; a_2 = 1$

Binomial array has no side lobes if $d \leq \lambda/2$

Dolph-Chebyshev Arrays:

A Compromise between uniform and binomial arrays.

AF can be represented as a Summation of Cosine terms. The largest harmonic Cosine term is one less than no. of elements.

$$
\begin{aligned}
\cos(mu) &= 1 && \text{for } m=0 \\
&= \cos u && m=1 \\
&= \cos 2u = 2\cos^2 u - 1 && m=2 \\
&= \cos 3u = 4\cos^3 u - 3\cos u && m=3 \\
&\ \vdots
\end{aligned}
$$

$$\cos(9u) = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u.$$

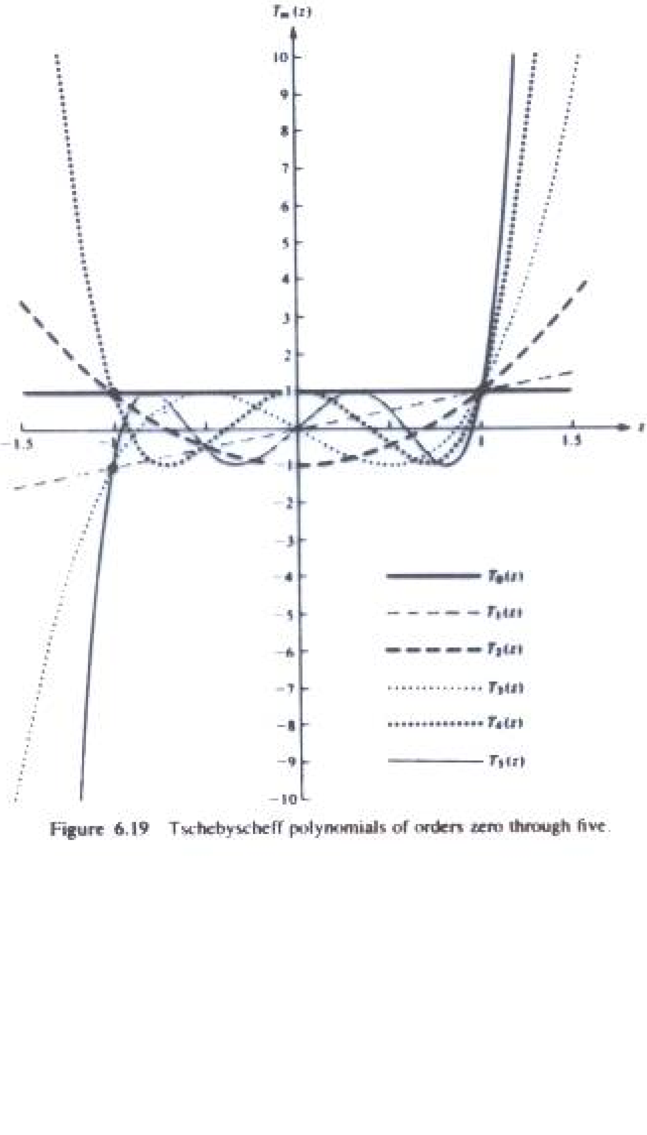


Figure 6.19 Tschebyscheff polynomials of orders zero through five.

Let $Z = \cos u$

m	$\cos(mu)$
0	$\cos(0) = 1 = T_0(z)$
1	$\cos(u) = Z = T_1(Z)$
2	$\cos(2u) = 2Z^2 - 1 = T_2(Z)$
3	$\cos(3u) = 4Z^3 - 3Z = T_3(Z)$
4	$\cos(4u) = 8Z^4 - 8Z^2 + 1 = T_4(Z)$
5	$\cos(5u) = 16Z^5 - 20Z^3 + 5Z = T_5(Z)$
⋮	
9	$\cos(9u) = 256Z^9 - 576Z^7 + 432Z^5 - 120Z^3 + 9Z = T_9(Z)$

Relationship valid only in $-1 \leq Z \leq 1$ range.

$\therefore \quad |T_m(z)| \leq 1 \quad$ for $-1 \leq Z \leq 1$

Recursion formula $\longrightarrow$ $T_m(z) = 2Z\, T_{m-1}(z) - T_{m-2}(z)$

We can also compute $T_m(z)$ from:

$$T_m(z) = \cos\left[m \cos^{-1}(z)\right] \qquad |z| \leq 1$$

$$T_m(z) = \cosh\left[m \cosh^{-1}(z)\right] \qquad |z| > 1$$

Properties:

1. All polynomials pass through $(1, 1)$
2. for $|z| \leq 1$ $\;|T_m(z)| \leq 1$
3. All roots occur within $|z| \leq 1$ & all maxima & minima have values $+1$ & -1.

Array Design Procedure :

1. Select the appropriate array factor (even or odd)

$$AF_{2M} = \sum_{1}^{M} a_n \cos[(2n-1)u] \quad \text{or} \quad AF_{2M+1} = \sum_{1}^{M+1} a_n \cos[2(n-1)u]$$

2. Expand AF by replacing $\cos(mu)$ by its series expansion.

3. Determine $Z = Z_0$ such that $T_m(Z_0) = R_0$.

   where m = No. of elements $- 1$.

4. Substitute $\cos(u) = \frac{Z}{Z_0}$ in AF

5. Equate the array factor to $T_m(Z)$.

6. Determine the excitation coefficients a_n's .

7. Write down the array factor .

Example Design a Dolph-Chebyshev array with $N = 10$, spacing between elements $= d$, $R_0 = -26\,dB$. Obtain the excitation coefficients & AF ,

$N = 10 \rightarrow$ we use $T_9(Z) = 256Z^9 - 576Z^7 + 432Z^5 - 120Z^3 + 9Z$

Then we use $(AF)_{10} = \sum_{1}^{5} a_n \cos[(2n-1)u] \qquad u = \frac{\pi d}{\lambda}\cos\theta$

Expanded AF $= a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u + a_4 \cos 7u + a_5 \cos 9u$.

Then, we replace $\cos 3u$, $\cos 5u$, $\cos 7u$, and $\cos 9u$ by their series expansion.

$$\therefore AF = a_1 \cos u + a_2 \left[4\cos^3 u - 3\cos u\right]$$
$$+ a_3 \left[16\cos^5 u - 20\cos^3 u + 5\cos u\right]$$
$$+ a_4 \left[64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u\right]$$
$$+ a_5 \left[256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u\right]$$

Calculate Z_0: $\quad R_0 = 26\,dB = 20\log_{10} R_0 \rightarrow R_0 = 20$

$$\therefore R_0 = 20 = T_9(z_0) = \cosh\left[9\cosh^{-1} z_0\right]$$

$$\text{or } Z_0 = \cosh\left[\tfrac{1}{9}\cosh^{-1}(20)\right] = 1.0851$$

We can also use: $Z_0 = \frac{1}{2}\left[\left(R_0 + \sqrt{R_0^2 - 1}\right)^{1/m} + \left(R_0 - \sqrt{R_0^2 - 1}\right)^{\frac{1}{m}}\right]$

Now we use the substitution $\cos u = \frac{Z}{1.0851} = \frac{Z}{Z_0}$ and equate it to $T_9(z)$

$$\therefore AF = \frac{Z}{Z_0}\left[a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5\right]$$
$$+ \frac{Z^3}{Z_0^3}\left[4a_2 - 20a_3 + 56a_4 - 120a_5\right]$$
$$+ \frac{Z^5}{Z_0^5}\left[16a_3 - 112a_4 + 432a_5\right]$$
$$+ \frac{Z^7}{Z_0^7}\left[64a_4 - 576a_5\right] + \frac{Z^9}{Z_0^9}\left[256a_5\right]$$
$$\equiv 9z - 120z^3 + 432z^5 - 576z^7 + 256z^9 \qquad (21)$$

Equating similar terms, we get:

$\therefore\ 256\, a_5 / z_0^9 = 256 \longrightarrow a_5 = 2.0860$

$\dfrac{64a_4 - 576a_5}{z_0^7} = -576 \longrightarrow a_4 = 2.8308$

$\dfrac{16a_3 - 112a_4 + 432a_5}{z_0^5} = 432 \longrightarrow a_3 = 4.1184$

$\dfrac{4a_2 - 20a_3 + 56a_4 - 120a_5}{z_0^3} = -120 \longrightarrow a_2 = 5.2073$

$\dfrac{a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5}{z_0} = 9 \longrightarrow a_1 = 5.8377$

If we normalize the coefficients:

$a_5 = 1$

$a_4 = 1.357$

$a_3 = 1.974$

$a_2 = 2.496$

$a_1 = 2.798$

$a_5 \quad \times \quad - - - a_1 \quad a_1 - - - - a_5$

$1 \quad - - - \quad 2.798 \quad 2.798 \quad - - \quad 1$

Amplitude taper.

The array factor can be written as:

$$AF = 2.798 \cos u + 2.496 \cos 3u + 1.974 \cos 5u + 1.357 \cos 7u + \cos 9u$$

where $u = \dfrac{\pi d}{\lambda} \cos\theta$

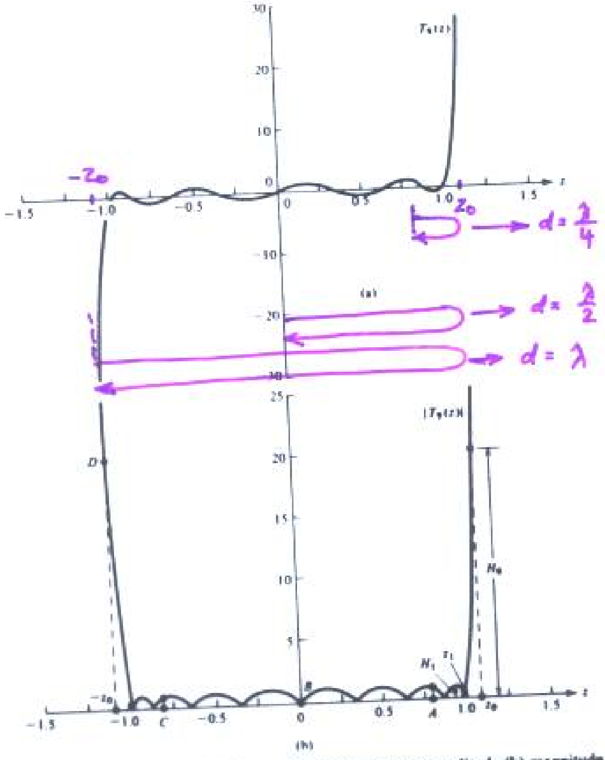


Figure 6.20 Tschebyscheff polynomial of order nine (a) amplitude (b) magnitude.

$$Z = Z_0 \cos u = Z_0 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right) = 1.0851 \cos\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

1. $d = \frac{\lambda}{4}$ $\quad \theta = 0^\circ \rightarrow Z = 0.7673$, $\theta = 90^\circ \rightarrow Z = Z_0$

$\theta = 180^\circ \rightarrow Z = 0.7673$