Kino-Dynamic, Harmonic, Potential Field- based Motion Planning Using Nonlinear Anisotropic Damping Forces

Ahmad A. Masoud
Electrical Engineering Department, KFUPM, P.O. Box 287, Dhaharan 31261, Saudi Arabia, masoud@kfupm.edu.sa

Abstract-*This paper extends the capabilities of the harmonic potential field approach to planning to cover both the kinematic and dynamic aspects of a robot's motion. The suggested approach converts the gradient guidance field from a harmonic potential to a control signal by augmenting it with a novel type of damping forces suggested in this paper called: nonlinear, anisotropic, damping forces (NADFs). The combination of the two provides a signal that can both guide a robot and effectively manage its dynamics. The kinodynamic planning signal inherits, fully, the guidance capabilities of the harmonic gradient field. It can also be easily configured to efficiently suppress the inertia-induced transients in the robot's trajectory without compromising the speed of operation. The approach works with dissipative systems as well as systems acted on by external forces without needing full knowledge of the system's dynamics. Theoretical developments and simulation results are provided in the paper.*

I. Introduction

The harmonic potential field approach to planning is emerging as a powerful paradigm for the guidance of autonomous agents. Since it was suggested in the mid-late eighties [1,2] the approach is continuously being developed to meet the stringent requirements operation in a real-life environment imposes on an agent. Until now, the approach has amassed many attractive properties crucial for enhancing goal reachability. The approach is provably-correct driving the agent to a successful conclusion if the task is manageable and providing an indication if the task is intractable. It can be used to guide the motion of an arbitrarily shaped agent in an unknown environment regardless of its geometry or topology relying only on the sensory data acquired online by the agent's finite range sensors. The method can also impose a variety of constraints on the agent's trajectory such as regional avoidance and directional constraints [3-8]. Harmonic functions are also Morse functions and a general form of the navigation functions suggested in [13], see appendix-1.

A planner may be defined as an intelligent, purposive, context-sensitive controller that can instruct an agent on how to deploy its motion actuators (i.e.generate a control signal) so that a target state may be reached in a constrained manner. Traditionally, a planning task is distributed on two stages: a high level control (HLC) stage and a low level control (LLC) stage. The HLC stage receives data about the environment, the target of the agent, and constraints on its behavior. It then simultaneously processes these data to generate a reference plan or trajectory marking the desired behavior of the robot. This trajectory, if actualized, leads to the agent reaching its target in the specified manner. The reference trajectory is then fed to an LLC in order to convert it into a sequence of action instructions to be executed by the agent's actuators of motion. Unfortunately, the HLC-LLC paradigm for planning suffers from serious problems that adversely impact its performance in a realistic setting. An alternative may be achieved by fusing the HLC and LLC modules into one called the navigation control. A navigation control attempts to directly convert the environmental data, goal of the robot, and constraints on its behavior into a control signal. Khatib potential field approach may be considered as one of the first methods to cast planning in a navigation control framework [9]. The potential field approach enjoys several attractive features; most significant is the high speed by which a robot can respond to the contents of its environment.

The attractor-repeller setting used to generate the potential field has some problems. The most serious one has to do with convergence where it was observed that a robot guided by such a method may stop somewhere in the workspace before reaching its target; the problem was termed the local minima problem. Many methods were proposed to generate potential fields that do not suffer from this problem [10-12]. Koditschek diffeomorphism approach [13] was among the first methods suggested to remedy this shortcoming. To convert the gradient guidance field from the potential surface ($-\nabla V$) into a control signal (u), Rimon et al. suggested that the gradient guidance field be augmented with a viscous dampening force that is linearly proportional to speed [14]:

$$u = -b \cdot \dot{x} - \nabla V(x) \qquad (1)$$

According to [14], this combination will only work provided that the initial speed of the robot at each point in space ($\omega(x)$) is lower than an upper bound S(x):

$$\omega(x) \le S(x) \qquad x \in \Omega \qquad (2)$$

where Ω is the workspace of the robot. Practical application of the above faced two difficulties: first, no method was provided to compute the upper bound S. Even if a method is devised for doing so, there is no guarantee that in a practical situation the initial speed of a robot can be made to lie below the admissible upper bound. The second difficulty has to do with the fact that the satisfaction of the upper speed constraint guarantees only that obstacle avoidance constraints will be upheld and convergence to the target will be achieved. In potential field methods, transients can be a serious concern that could make it impractical to use these techniques for controlling a robot. Also, the approach seems to only deal with dissipative systems where no mention of how the method may be applied when external forces such as gravity are present.

In its current form the harmonic potential field (HPF) approach can only operate in an HLC mode providing only a guidance signal from the gradient of the potential. This signal has to be converted into a control signal by an LLC. Guldner and Utkin suggested an interesting approach based on a sliding mode control for converting the gradient field from an HPF into a control signal [21]. The approach is robust, has good convergence properties, does not require full knowledge of system dynamics and can make, with little transients, the dynamic trajectory of the robot follow the kinematic trajectory marked by the gradient field. The main drawback of the approach seems to be the high chattering the control signal experiences.

In this paper a method is suggested to utilize the HPF approach in a navigation control mode by augmenting the gradient guidance field from an HPF with a new type of damping force called: nonlinear anisotropic damping forces (NADFs). It is shown that an NADF-based control can efficiently suppress inertia-induced artifacts in the dynamical trajectory of the system making it closely follow the kinematic trajectory while maintaining an agile system response. The approach does not require the system dynamics to be fully known. A loose upper bound is sufficient for constructing a well-behaved control signal that can deal with dissipative systems as well as systems being influenced by external forces (e.g. gravity). Earlier version of this work may be found in [32].

This paper is organized as follows: section II provides a brief background of the potential field approach. The NADF approach is presented in section III. Sections IV and V discuss the application of the approach to dissipative systems and systems experiencing external forces respectively. Simulation results are in section VI, and conclusions are placed in section VII.

II. Background

The HPF approach appeared shortly after the work of Khatib. Although the approach was brought to the forefront of motion planning independently and simultaneously by different researchers [16-20], the first work to be published on the subject was that by Sato in 1987 [1]. The HPF approach eliminates the local minima problem encountered in [9] by forcing the differential properties of the potential field to satisfy the Laplace equation inside the workspace of the robot (Ω) while constraining the properties of the potential at the boundary of Ω ($\Gamma=\partial\Omega$). The boundary set Γ includes both the boundaries of the forbidden zones (O) and the target point (x_T). A basic setting of the HPF approach is:

$$\nabla^2 V(x) \equiv 0 \qquad x\in\Omega$$

subject to: $$V_i = 0|_{x=C_i} \;\&\; V_i = 1|_{x\in\Gamma_i^-} \;. \tag{3}$$

The trajectory to the target (x(t)) is generated using the HPF-based, gradient dynamical system:

$$\dot{x} = -\nabla V(x) \qquad x(0) = x_0 \in\Omega \tag{4}$$

The trajectory is guaranteed to:

$$- \lim_{t\to\infty} x(t) \to x_T \qquad - \; x(t)\in\Omega \qquad \forall t \tag{5}$$

whereby a proof of (5) may be found in [3]. Figure-1 shows the negative gradient field of a harmonic potential and the trajectory, x(t), generated using the gradient dynamical system in (4) for the simple environment of a room with two dividers. It ought to be mentioned that the HPF approach is only a special case of a broader class of planners called PDE-ODE motion planners [5] where the field is generated using the boundary value problem:

solve: $\qquad L(V(x)) \equiv 0 \quad x\in\Omega$

subject to: $\qquad \Psi(V(x)) = 0 \quad x\in\Gamma. \qquad (6)$

The trajectory is generated using the nonlinear system:

$$\dot{x} = F(V(x)) \qquad x(0) = x_0 \in\Omega \tag{7}$$

where L is scalar partial differential operator, Ψ is a governing relation restricting the potential or some of its properties at the boundary to a certain value, F is a nonlinear vector function mapping $R\to R^N$, N is the dimension of x, PDE stands for partial differential equation, and ODE stands for ordinary differential equation. Planners assuming a PDE-ODE setting other than that of the one in (3) may be found in [3,7,8].

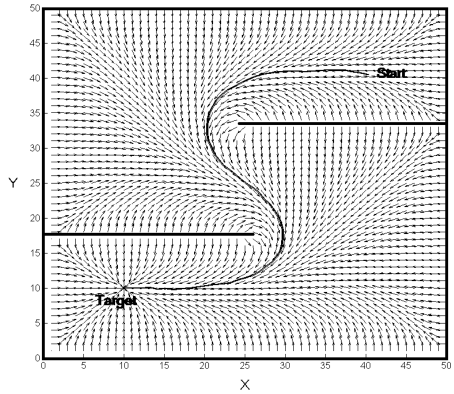

Figure-1: Guidance field and generated trajectory of an HPF.

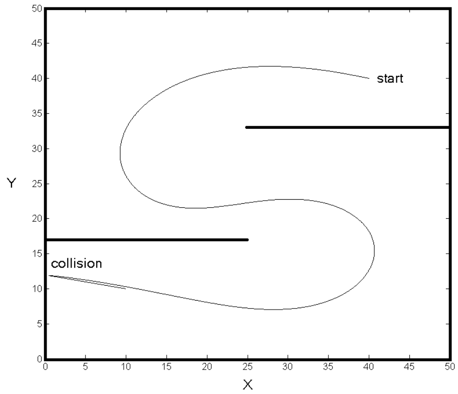

Figure-2: trajectory of a point mass controlled by the field in figure-1.

The trajectory, x(t), generated by the dynamical system in (4) is only a reference trajectory that should be fed to an LLC in order to generate the control signal, u. One way of converting the guidance signal into a control signal is to augment the gradient field with a component that is proportional to speed. This seemingly straightforward solution is problematic. In figure-2, the negative gradient of the potential in figure-1 is used to navigate a 1kg point mass. The system equation is:

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = -b\cdot\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} - \begin{bmatrix}\partial v(x,y)/\partial x\\ \partial v(x,y)/\partial y\end{bmatrix} \tag{8}$$

where b=0.1. Despite the initial speed being zero, the trajectory violated the avoidance condition and collided with the wall.

III. The NADF Approach

An intuitive solution for converting a gradient guidance field into a navigation control signal is to increase the coefficient of the linear velocity term to a sufficiently high level. The linear velocity component acts as a dampener of motion that may be used to place the inertial force under control by marginalizing its disruptive influence on the trajectory of the robot that the gradient field is attempting to generate. The following example demonstrates that this solution is impractical. In order to generate a control signal that would satisfy the avoidance constraints (5), the coefficient of damping of the system is increased to b=0.15. Figure-3 shows the resulting trajectory and figure-4 shows the distance to the target as a function of time. Although the trajectory did converge to the target point (x_T) and did not violate the regional avoidance constraints, unacceptable transients along with significant deviations from the path marked by the gradient field (figure-1) are present. In a second attempt to generate a well-behaved control signal, the dampening coefficient is significantly increased to b=.7.

Although a well-behaved trajectory was obtained (figure-5), significant slowdown of motion did occur (figure-6).

The method for converting the gradient field from a harmonic potential into a navigation control signal by simple augmentation with a linear velocity damping term is incorrect. This approach ignores the dual role the gradient field plays as a control and guidance provider. The field guides a robot to the target using vectors that point out the directions along which the robot has to move if the target is to be reached and the obstacles are to be avoided. At the same time, these vectors are forces that act on the mass of the robot in order to actuate motion. The inertia of the robot will have a disruptive influence on motion. The linear damping term manages the inertial forces in an attempt to make the motion yield to the guidance provided by the gradient field. A damping component that is proportional to velocity exercises omni-directional attenuation of motion regardless of the direction along which it is heading. This means that the useful component of motion marked by the direction along which the goal component of the gradient of the potential is pointing is treated in the same manner as the unwanted inertia-induced component of the trajectory. These two components should not be treated equally. Attenuation should be restricted to the inertia-caused disruptive component of motion, while the component in conformity with the guidance of the artificial potential should be left unaffected (figure-7).

To better manage the effect of the inertial forces, a damping component that treats the gradient of the artificial potential both as an actuator of dynamics and as a guiding signal is needed. A damping force (h) that behaves in the above manner is:

$$h(x,\dot{x}) = [(\mathbf{n}^t \dot{x}\,\mathbf{n} + (\frac{\nabla V(x)^T}{\|\nabla V(x)\|}\cdot \dot{x}\cdot \Phi(\nabla V(x)^T \dot{x}))\frac{\nabla V(x)}{\|\nabla V(x)\|}] \qquad (9)$$

where $\mathbf{n}$ is a unit vector orthogonal to ∇V and Φ is the unit step function. This force is given the name: nonlinear, anisotropic, damping force (NADF). For the two dimensional case, an NADF has the form:

$$h = \frac{1}{g_x^2 + g_y^2}\left[(g_x\dot{y} - g_y\dot{x})\cdot\begin{bmatrix}-g_y\\ g_x\end{bmatrix} + (g_x\dot{x} + g_y\dot{y})\cdot\Phi(-g_x\dot{x} - g_y\dot{y})\begin{bmatrix}g_x\\ g_y\end{bmatrix}\right] \qquad (10)$$

where $\nabla V(x,y) = [g_x \quad g_y]^T$. A procedure for computing the component of motion normal to $-\nabla V$ in R^N is in appendix-2.

IV- Dissipative Systems

In this section two propositions are stated and proven. The first proposition shows that a gradient field of a harmonic potential generated by the boundary value problem in (3) combined with NADF can guarantee global, asymptotic convergence of a fully actuated second order dissipative dynamical system. The second shows that the dynamic trajectory of the system can be made arbitrarily close to the kinematic trajectory generated by the system in (4); hence, preserving the spatial constraints.

Proposition-1: Let V(x) be a harmonic potential generated using the boundary value problem in (3). The trajectory of the dynamical system:

$$D(x)\ddot{x} + C(x,\dot{x})\dot{x} = u \qquad (11)$$

$$u = -b_d \cdot h(x,\dot{x}) - k\cdot\nabla V(x)$$

will globally, asymptotically converge to:

$$\lim_{t\to\infty}\begin{bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{bmatrix} \to \begin{bmatrix}\mathbf{x_T}\\ \mathbf{0}\end{bmatrix} \qquad (12)$$

for any positive constants b_d and k, where $x\in R^N$, $V(x):R^N\to R$, D(x) is an N×N positive definite inertia matrix, $C(x,\dot{x})\dot{x}$ contains the centripetal, Coriolis, and gyroscopic forces. Proof of the above proposition is carried out using the LaSalle principle [23].

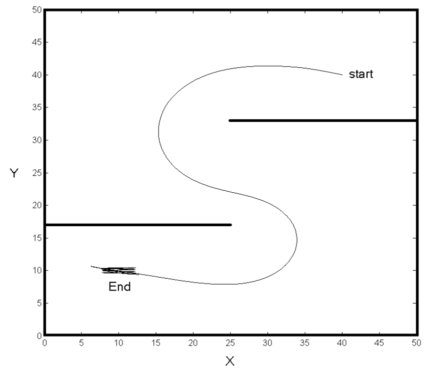

Figure-3: Trajectory, point mass, linear damping increased.

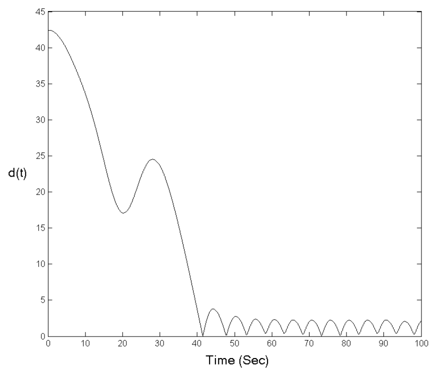

Figure-4: Distance to target versus time.

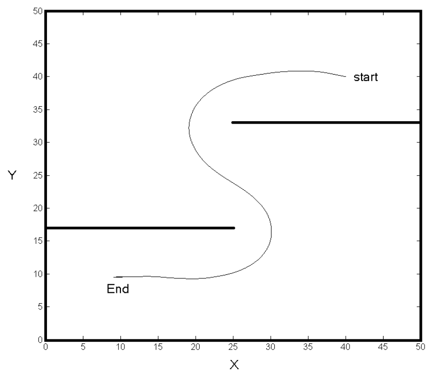

Figure-5: Trajectory, high linear damping.

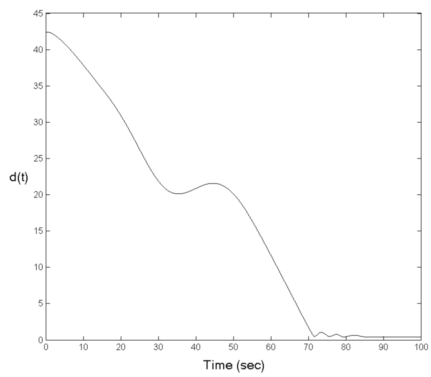

Figure-6: Distance to target versus time.

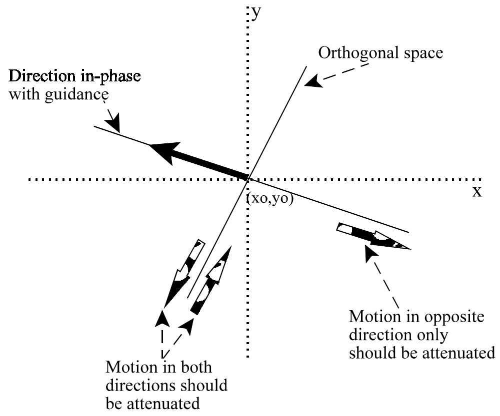

Figure-7: nonlinear, anisotropic, damping force (NADF).

Proof: Let Ξ be the Liapunov function candidate:

$$\Xi(x,\dot{x}) = k\cdot V(x) + \frac{1}{2}\dot{x}^T D(x)\dot{x} \qquad (13)$$

Note that since V(x) is harmonic, it must assume its maxima on Γ and minima on x_T . In other words, V(x) can only be zero at x_T; otherwise, its value is greater than zero:

$$\Xi(x,\dot{x}) = \begin{cases} 0 & \text{iff} \quad x = x_T, \dot{x} = 0 \\ \text{positive} & \text{otherwise} \end{cases} . \qquad (14)$$

The time derivative of the above function is:

$$\dot{\Xi}(x,\dot{x}) = k\cdot\nabla V(x)^T\dot{x} + \frac{1}{2}\dot{x}^T\dot{D}(x)\dot{x} + \dot{x}^T D(x)\ddot{x} \ . \qquad (15)$$

Substituting:

$$\ddot{x} = D^{-1}(x)[-C(x,\dot{x})\dot{x} - b_d\cdot h(x,\dot{x}) - k\cdot\nabla V(x)] \qquad (16)$$

along with (9) in the above equation yields:

$$\begin{aligned} \dot{\Xi} \ = \ & k\cdot\nabla V(x)^T\dot{x} + \frac{1}{2}\dot{x}^T\dot{D}(x)\dot{x} \\ & - k\cdot\nabla V(x)^T\dot{x} - \dot{x}^T C(x,\dot{x})\dot{x} \\ & - b_d\cdot\dot{x}^T(n^T\dot{x})\mathbf{n} \\ & - b_d\cdot\dot{x}^T\left(\frac{\nabla V(x)^T}{|\nabla V(x)|}\cdot\dot{x}\cdot\Phi(-\nabla V(x)^T\dot{x})\frac{\nabla V(x)}{|\nabla V(x)|}\right) \end{aligned} \qquad (17)$$

Using the passivity property:

$$\dot{x}^T(\dot{D}(x) - 2\cdot C(x,\dot{x}))\dot{x} = 0 \qquad (18)$$

and rearranging the terms we get:

$$\begin{aligned} \dot{\Xi} \ = \ & -b_d\cdot(n^T\dot{x})^T(n^T\dot{x}) \\ & -b_d\cdot\frac{(\nabla V(x)^T\cdot\dot{x})^T}{|\nabla V(x)|}\cdot\frac{(\nabla V(x)^T\cdot\dot{x})}{|\nabla V(x)|}\cdot\Phi(\nabla V(x)^T\dot{x}) \end{aligned} \qquad (19)$$

as can be seen $\quad \dot{\Xi} \le 0 \quad \forall x, \dot{x}$, (20)

where $\quad \dot{\Xi} = 0 \quad$ for $\ \forall x \in \Omega, \dot{x} = 0$

according to LaSalle principle any bounded solution of (11) will converge to the minimum invariant set:

$$E \subset \{\dot{x} = 0, x\} \ . \qquad (21)$$

Determining E requires studying the critical points of V(x). According to the maximum principle, x_T is the only minimum (stable equilibrium point) V(x) can have. Besides x_T , V(x) has a finite number of isolated critical points $\{x_i\}$ at which $\nabla V=0$; however, the hessian at these points is non-singular, i.e. V(x) is Morse [24]. A proof of this result may be found in appendix-1. From the above it is concluded that E contains only one point, $x = x_T, \dot{x} = 0$, to which motion will converge. A proof based on Liapunov theory showing that, for the kinematic case, $-\nabla V(x)$ can drive motion from anywhere in Ω to x_T may be found in [3].

Proposition-2: Let ρ be the trajectory constructed as the spatial projection of the solution, x(t), of the first order differential system in (4). Also let ρ_d be the trajectory constructed as the spatial projection of the solution, x(t), of the second order system in (11), figure-8. Then there exist a b_d that can make the maximum deviation (δ_m) between ρ and ρ_d arbitrarily small.

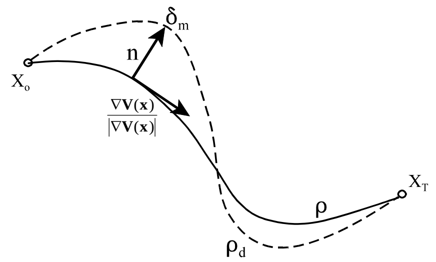

Figure-8: The kinematic and dynamic trajectories.

Proof: The gradient field from an HPF does not only work as a guide of motion to the target; it also may be used to cover Ω with a complete set of boundary-fitted basis [4] coordinates.

The radial basis of the system ($\nabla V/|\nabla V|$) marks the useful component of motion. The basis orthogonal to this component spans the instantaneous deviation between ρ and ρ_d (δ) which NADF is required to attenuate (figure-9).

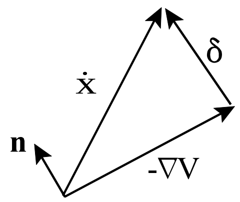

Figure-9: The disruptive component of motion.

The dynamic equation describing the disruptive component is:

$$\mathbf{n}^T D(x)\ddot{x} + \mathbf{n}^T C(x,\dot{x})\dot{x} + b_d\cdot\mathbf{n}^T h(x,\dot{x}) + k\cdot\mathbf{n}^T\nabla V(x) = 0 \qquad (22)$$

Examining the above equation term by term yields:

1- $\quad \mathbf{n}^T\nabla V = 0$, (23)

2- $\quad \mathbf{n}^T\left[(n^t\dot{x})n + \left(\frac{\nabla V(x)^T}{|\nabla V(x)^T|}\cdot\dot{x}\cdot\Phi(\nabla V(x)^T\dot{x})\right)\frac{\nabla V(x)}{|\nabla V(x)|}\right] = (\mathbf{n}^t\dot{x})$

3- assuming a stable and non-impulsive system, an upper bound can be placed on the speed: $\quad |\dot{x}| \le \nu_{max}$. (24)

Therefore, the norm of the matrix C may be bound as:

$$\|\mathbf{C}(x,\dot{x})\| \le c_{max} \qquad (25)$$

4- any inertia matrix belonging to a physical system is positive definite, invertible, and have a bounded norm:

$$\|D(x)\| \le d_{max} \qquad (26)$$

where d_{max}, c_{max}, and ν_{max} are finite, positive constants. A dynamic equation that yields an upper bound on δ is:

$$d_{max}\cdot\mathbf{n}^T\ddot{x} - c_{max}\mathbf{n}^T\dot{x} + b_d\cdot\mathbf{n}^T\dot{x} = 0 \qquad (27)$$

or $\quad \ddot{\delta} + \Delta\cdot\dot{\delta} = 0$

where $\quad \ddot{\delta} = \mathbf{n}^T\ddot{x}$, $\dot{\delta} = \mathbf{n}^T\dot{\mathbf{x}}$, and $\quad \Delta = \frac{b_d - c_{max}}{d_{max}}$.

To determine the effect of the disruptive time component ($\xi(t)$) that acts normal to ∇V, the impulse response (z(t)) of (27) is obtained:

$$z(t) = \frac{1}{\Delta}(1 - \mathbf{e}^{-\Delta\cdot t})\Phi(\mathbf{t}) = \frac{\grave{z}(t)}{\Delta} \ . \qquad (28)$$

The deviation as a function of time may be computed as:

$$\delta(\mathbf{t}) = \xi(\mathbf{t}) * z(t)$$

where * denotes the convolution operation. Since it was shown in proposition-1 that motion will converge to x_T and all dynamic terms will tend to zero, $\xi(t)$ may be bounded as:

$$\int_0^\infty |\xi(t)| dt \le I \quad , \tag{29}$$

therefore: $\quad \delta(t) = \dfrac{\mathbf{1}}{\Delta} z`(t) * \xi(t) \le \dfrac{I_{max}}{\Delta}$,

where I and I_{max} are positive constants. By properly selecting a value for Δ, the maximum deviation δ_m can be made arbitrarily small. In other words the dynamic trajectory of (11) will closely follow the kinematic trajectory of (4) and the spatial constraints will be preserved. It ought to be mentioned that since NADF is by design made to be zero when motion is in accordance with the guidance field ∇V, b_d can be made arbitrarily large without slowing down the system. This fact is clearly reflected by the simulation results (figure-15).

V. Systems with External Forces

The NADF approach may be adapted for designing constrained motion controller for mechanical systems experiencing external forces (e.g. gravity). The dynamical equation of such systems has the form:

$$D(x)\ddot{x} + C(x,\dot{x})\dot{x} + g(x) = F \tag{30}$$

where g(x) and F are vectors containing the external forces and the applied control forces respectively. A controller consisting of the gradient guidance field and a strong enough NADF (31) has the ability to make the trajectory of the system in (30) closely follow the kinematic trajectory from an initial starting point (x_o) to the target point x_T,

$$F = -b_d \cdot h(x,\dot{x}) - k \cdot \nabla V(x) \cdot \tag{31}$$

However, due to the presence of the external forces the controller will not be able to hold the state close to the target point and drift will occur (Figure-16). Arimoto and Miyazaki showed that steady state error caused by the external forces may be cancelled by using an integral control action [27]. Unfortunately, an integral action raises the order of the mechanical system and could cause it to become unstable if it is not tuned properly. The integrator also induces a difficult to manage transients in the system response.

Here an alternative approach to using an integrator is suggested. The suggested approach does not endanger stability and can cancel the error caused by the external forces bringing the dynamic trajectory arbitrarily close to the target point. The approach capitulates on the ability of the controller in (31) to drive motion arbitrarily close to the target point. Once the trajectory is close to the target, a passive clamping control action is activated to trap the trajectory in a set close to the target. After motion is trapped by the clamping control, an iterative procedure is suggested for totally cancelling the error. In the following the suggested clamping control is described.

1. Clamping control:

The effect of the clamping control (F_c) is strictly localized to a hyper-sphere of constant radius σ surrounding the target point. If motion is heading towards the target, this control component is inactive. On the other hand, if motion starts heading away from the target, the control becomes active and attempts to drive the trajectory back to the target (Figure-10).

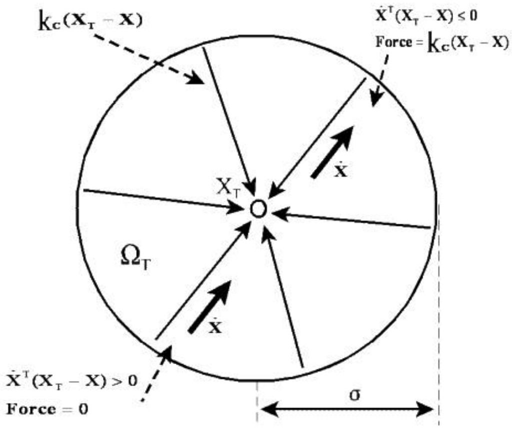

Figure-10: The clamping control.

A clamping control that behaves in the above manner is:

$$\Phi_C(x,\dot{x}) = (x - x_T)\cdot\Phi(\sigma - |x - x_T|)\cdot\Phi(\dot{x}^T(x - x_T)) \tag{32}$$

The strength of F_c is adjusted by multiplying it with a constant k_c so that the steady state error is kept below a desired level (ϵ). Unlike the integrator, the use of a clamping control will keep the mechanical system stable for any positive value of k_c.

Proposition-3: For the mechanical system in (30), a controller of the form:

$$F = -b_d \cdot h(x,\dot{x}) - k\cdot\nabla V(x) - k_C \cdot F_C(\dot{x}, x) \tag{33}$$

can make $\quad \lim_{t\to\infty} |x(t) - x_T| \le \varepsilon < \sigma$

and $\quad \lim_{t\to\infty} \dot{x} = 0 \quad (34)$

provided that:

1- k, b_d, and k_c are all positive,

2- $\quad k_c \ge F_{max}/\epsilon$,

$$F_{max} = \max_X |g(x)| \qquad x\in\Omega_\sigma$$

and $\quad \Omega_\sigma = \{x : |x - x_T| \le \sigma\}$. (35)

3- a high enough value of b_d is selected so that at some instant in time t\`

$$|x(t`) - x_T| < \sigma \tag{36}$$

4- k is high enough so that the gradient field is capable of directing the trajectory to Ω_σ

$$|k\cdot\nabla V(x)| > \left| g^T(x)\frac{\nabla V(x)}{|\nabla V(x)|}\right| \quad x\in\Omega\text{-}\Omega_\sigma \tag{37}$$

Proof: Consider a Liapunov function candidate similar to the one in (13) with a gravitational potential energy term (P(x)) added: $\quad \Xi(x,\dot{x}) = k\cdot V(x) + \dfrac{1}{2}\dot{x}^T D(x)\dot{x} + P(x) \quad (38)$

note that: $\quad g(x) = -\nabla P(x)$ and $P(x) = \displaystyle\int_{x_0}^{x} g(z)dz$. (39)

Differentiating (38) with respect to time we get:

$$\dot{\Xi}(x,\dot{x}) = k\cdot\nabla V(x)^T\dot{x} + \frac{1}{2}\dot{x}^T\dot{D}(x)\dot{x} + \dot{x}^T D(x)\ddot{x} + \dot{x}^T g(x) \tag{40}$$

solving for $\ddot{x}$ from equations (30, 31) and substituting the results in (40) we get:

$$\begin{aligned}\dot{\Xi} \;\; = \;\; & -b_d\cdot(n^T\dot{x})^T(n^T\dot{x}) \\ & - b_d\cdot\frac{(\nabla V(x)^T\cdot\dot{x})^T}{|\nabla V(x)|}\cdot\frac{(\nabla V(x)^T\cdot\dot{x})}{|\nabla V(x)|}\cdot\Phi(\nabla V(x)^T\dot{x}) \\ & - k_C\cdot\dot{x}^T(x - x_T)\cdot\Phi(\dot{x}^T(x - x_T))\cdot\Phi(\sigma - |x - x_T|)\end{aligned} \tag{41}$$

Since k_c and b_d are positive we have:

$$\dot{\Xi} \le 0 \qquad \forall x, \dot{x}\,, \quad \text{where}$$
$$\dot{\Xi} = 0 \qquad \text{for} \quad \forall x, \dot{x} = 0\ . \tag{42}$$

Since we are assuming that k and b_d are selected high enough so that the dynamic trajectory will follow the kinematic trajectory and enter Ω_σ , the minimum invariant set to which the trajectory is going to converge may be computed from:

$$g(x) + k \cdot \nabla V(x) + k_C \cdot F_C(x, \dot{x} = 0) = 0 \tag{43}$$

Since $\Phi(0)$=1, and $x \in \Omega_\sigma$ (i.e. $\Phi(\sigma-|x-x_T|)$=1), equation (43) becomes:

$$g(x) + k \cdot \nabla V(x) + k_C \cdot (x - x_T) = 0 \tag{44}$$

As can be seen if condition 2 on k_C is satisfied, the solution of the above equation has to lie in the set $\Omega_\epsilon = \{x: |x-x_T| < \epsilon\}$. This means that the deviation of the end of the dynamic trajectory from the target point should at most be ϵ.

Another alternative to the use of integration is to reduce steady state error by increasing the gain of the gradient field (k) to a sufficiently high level. This approach makes the transient difficult to manage and increases the control effort. On the other hand, selecting a high gain of the clamping control (k_C) to manage the steady state error will not cause the above problems. This is due to the fact that this control component is designed to be minimally intrusive affecting the system only when it is needed. This is clearly demonstrated by simulation (figure-17,18)

2. Iterative, blind error cancellation:

While clamping control has the ability to reduce the steady state error to an arbitrarily small value, sometimes it is desired that this error be totally cancelled. Here, an iterative, blind procedure is suggested for error cancellation. The procedure works by providing an alternative path (β) other than the error channel ($K_P \cdot e$, where K_P is a positive definite matrix) to supply the control signal (u) that is needed to hold the robot at a location x_T (figure-11),

$$u = K_P \cdot e + \beta \tag{45}$$

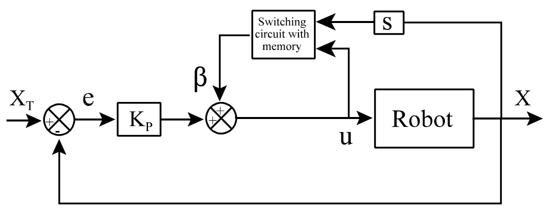

Figure-11: The suggested scheme for iterative error cancellation.

The fixed point iteration method [28] is used to evolve an estimate of the control signal so that the steady state error is driven to zero. This procedure is implemented using a switched logic circuit with one memory storage element. One implementation requires the circuit to have two inputs: the control that is directly fed to the robot and velocity of the robot's coordinates in order to assess convergence (other means to decide if the robot has converged may be used). There is only one output consisting of the bias term β. The bias term is iterativly determined as follows: when motion is about to settle (i.e. $|dx/dt| < \alpha$, where $0 < \alpha << 1$), the circuit measures the value of u and assigns it to β. This value is kept till at another instant i the event becomes true again. At the i'th instant we have:

$$u = g(x_i),\ \ \beta = g(x_{i-1}),\ \text{and}\ \ K_P \cdot e = K_P \cdot (x_T - x_i) \tag{46}$$

where x_i is the position of the robot at the i'th settling instant. Relating the above quantities using (45) yields the recursive relation:

$$g(x_i) = g(x_{i-1}) + K_P \cdot (x_T - x_i)\ . \tag{47}$$

Proposition-4: The recursive relation in (47) has a fixed point at which

$$(x_T - x_i) = 0\ . \tag{48}$$

Proof: Using Taylor series expansion around x_T, we have:

$$g(x) = g(x_T) + J(g(x_T))(x-x_T) + \ldots = g(x_T) + F(x-x_T) \tag{49}$$

where J is the Jacobian matrix of g and F is a function containing the $(x-x_T)$ terms of the Taylor series. Substituting (49) into (47) we get:

$$F(e`_i) = F(e`_{i-1}) - K_P \cdot e`_i \tag{50}$$

where

$$e`_i = -(x_T - x_i)\ . \tag{51}$$

Now let η=F(e`) and Q be the inverse function of F in the neighborhood of x_T. Substituting Q in (50), we obtain the recursive relation:

$$K_P \cdot Q(\eta_i) + \eta_i = \eta_{i-1}\ . \tag{52}$$

At a fixed point we have : $\eta_i = \eta_{i-1}$, or

$$K_P \cdot Q(\eta_i) = 0. \tag{53}$$

Since K_P is positive definite, i.e. it is not singular:

$$Q(\eta_i) = e`_i = (x_i - x_T) = 0 \tag{54}$$

In other words: $x_i = x_T$.

Proposition-5: For any positive definite K_P, the fixed point $x=x_T$ is a stable attractor, i.e. if x_i is sufficiently close to x_T,

$$\lim_{i\to\infty} x_i \to x_T \tag{55}$$

Proof: In the close neighborhood of x_T, equation (47) may be written as:

$$J(g(x_T)) \cdot (x_i - x_T) = J(g(x_T)) \cdot (x_{i-1} - x_T) + K_P \cdot (x_T - x_i) \tag{56}$$

Notice that: $J(g(x_T)) = J(\nabla P(x_T)) = H(x_T)$ (57)

where H is the symmetric hessian matrix. Substituting (57) in (56) yields the equation:

$$[K_P + H(x_T)] \cdot e_i = H(x_T) \cdot e_{i-1} \tag{58}$$

where

$$e_i = (x_T - x_i)\ .$$

Since K_P is positive definite and H is symmetric, they are simultaneously diagonalizable into:

$$K_P = UU^T \text{ and } H = U\Lambda U^T \tag{59}$$

where U is a nonsingular matrix and Λ is a diagonal matrix with non-negative elements λ_l, l=1,..,N, see [29, page-86].Using the above decomposition (58) may be written as:

$$U(I+\Lambda)U^T \cdot e_i = U\Lambda U^T \cdot e_{i-1} \tag{60}$$

Using the transformation $q_i = U^T \cdot e_i$,

we have

$$q_i = A \cdot q_{i-1} \tag{61}$$

where

$$\mathbf{A} = (\mathbf{I} + \Lambda)^{-1}\Lambda = \begin{bmatrix} \frac{\lambda_1}{\mathbf{1}+\lambda_1} & \mathbf{0} & \cdot & \mathbf{0} \\ \mathbf{0} & \frac{\lambda_2}{\mathbf{1}+\lambda_2} & \cdot & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \cdot & \frac{\lambda_N}{\mathbf{1}+\lambda_N} \end{bmatrix}. \tag{62}$$

It is well-known that the solution of (61) is:

$$q_i = A^i \cdot q_0 \tag{63}$$

Since

$$0 \le \frac{\lambda_l}{1+\lambda_l} < 1 \qquad l=1,..,N \tag{64}$$

we have:

$$\lim_{i\to\infty} q_i = \lim_{i\to\infty} U^T \cdot e_i \to 0\,. \tag{65}$$

Since U is a nonsingular matrix :

$$\lim_{i\to\infty} e_i \to 0 \tag{66}$$

or

$$\lim_{i\to\infty} x_i \to x_T \tag{67}$$

VI. Results

1. Point mass in a cluttered environment:

The gradient field in figure-1 is augmented with NADF instead of the linear, viscous, damping forces. The combination of both gradient field and NADF is used to steer a 1Kg mass from a start point to a target point. An excessively high damping coefficient, b_d=10, is used. The trajectory of the mass is shown in figure-12. As can be seen, the kinodynamic trajectory of the mass is almost identical to that marked by the gradient field (kinematics only) in figure-1. Moreover, motion of the mass is almost six times faster than its viscous damping counterpart shown in figure-5 with a settling time (T_S) of about 12 seconds compared to 72 seconds. Figure-13 shows the control signal (X-Y force components).

2. Settling time - a comparison:

NADF and linear damping exhibit different behavior as far as convergence is considered. The settling time for the point mass with no constraints on speed example is drawn in figure-14 as a function of the linear viscous friction coefficient (b). As can be seen, the T_S-b relation is convex with one value for b corresponding to a global minimum of T_S. This is expected since for low b high oscillations will prevent motion from quickly settling in the 5% zone around the target. On the other hand, a high value for b reduces the oscillations by slowing down the response delaying the entrance to the 5% zone.

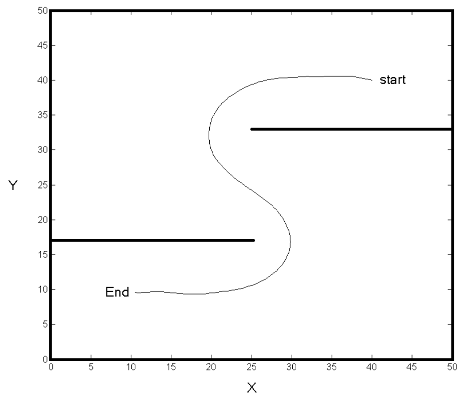

Figure-12: Trajectory, NADF, b_d=10.

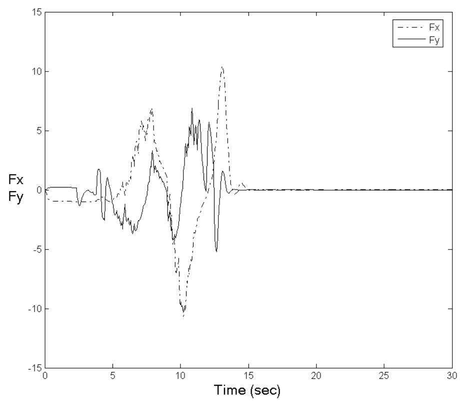

Figure-13: x and y control force components.

The relation between T_S and the coefficient of NADF (b_d) is a rapidly and strictly decreasing one **(**figure-15). Similar to the linear case, for a low value of b_d high oscillations will prevent the quick capture of the trajectory in the 5% zone around the target. As the value of b_d increases, NADF only impedes the component of motion along the coordinate field tangent to the gradient guidance field. This component does not contribute to convergence and only causes delay in reaching the target. Since NADF attenuates only this component of motion leaving the motion along the gradient field unaffected, the delay in reaching the target drops as b_d increases yielding a strictly decreasing profile of the T_S-b_d curve. The T_S versus the coefficient of damping profile is important. It determines the ability to tune the controller so that the specifications are met. In tuning the controller there are two requirements: it is required that the maximum spatial deviation (δ_m) between the kinematic and the dynamic paths be as small as possible so that the constraints are upheld. It is also required that the settling time be as small as possible. The first requirement is achieved by making the coefficient of damping high enough. In the linear viscous damping case one can only strike a compromise between T_S and $\boldsymbol{\delta}_m$. For the NADF case this compromise is not needed since both T_S and δ_m are strictly decreasing as a function of b_d.

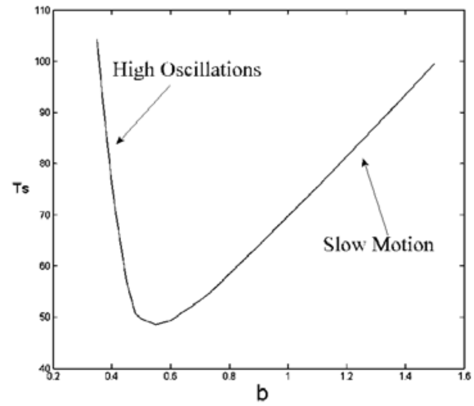

Figure-14: T_S versus b for linear damping.

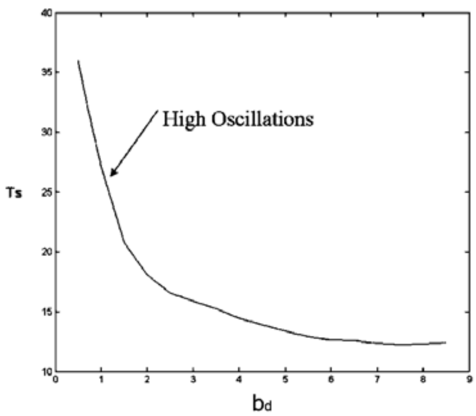

Figure-15: Settling time versus NADF coefficient.

3. Point mass with external forces

The NADF approach may be adapted to work with second order systems experiencing external forces using the suggested clamping control. In this example a point mass with constant external forces acting on it having the system equation in (68) is controlled using a gradient field and NADF.

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} Fx \\ Fy \end{bmatrix} \tag{68}$$

As can be seen from figure-16, for a sufficiently high b_d the controller will succeed in driving the mass to the target and avoiding the obstacles. However, when the target is reached, drift caused by the external forces occur.

In figure-17 a clamping control similar to the one in (32) is added with k=1, b_d=10, k_C=10. As can be seen, the controller was able to hold the trajectory near the target point relying only on a loose, upper bound estimate of the drift. Despite the high value of k_C, the trajectory settled in an overdamped manner. The Fx, Fy control forces are shown in figure-18.

The sliding mode (SM) control approach suggested by Guldner and Utkin in [21] for converting a gradient guidance signal into a control signal has the ability to handle systems with external forces. In this approach a sliding surface (γ) is defined as:

$$\gamma = \dot{x} - v_d(t)\frac{-\nabla V}{\|\nabla V\|} \quad . \tag{69}$$

The control signal is:
$$F = -F_0 \frac{\gamma}{\|\gamma\|} \tag{70}$$

where v_d and F_o are the maximum allowable speed and control forces respectively. The sliding mode control is applied to the point mass with drift in (68). The parameters of the sliding surface are set so that a settling time of 6 sec is obtained. F_o is set to obtain a maximum control effort of 100 N. The trajectory is shown in figure-19. The control forces are shown in figures-20,21. Compared to the NADF approach with clamping the trajectory obtained using the SM approach is a little shaky and experiences some oscillations near the target. However, the biggest difference has to do with the quality and magnitude of the control signals used by both approaches.

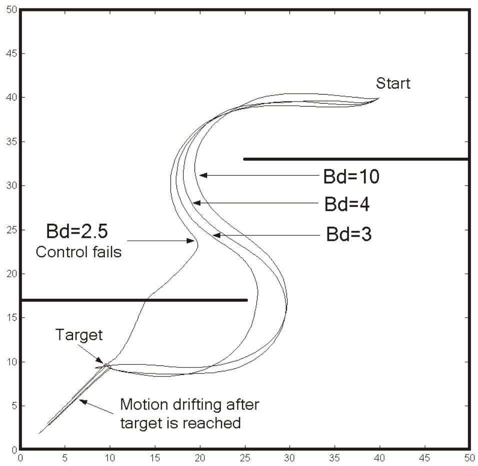

Figure-16: Trajectory, NADF - external force present.

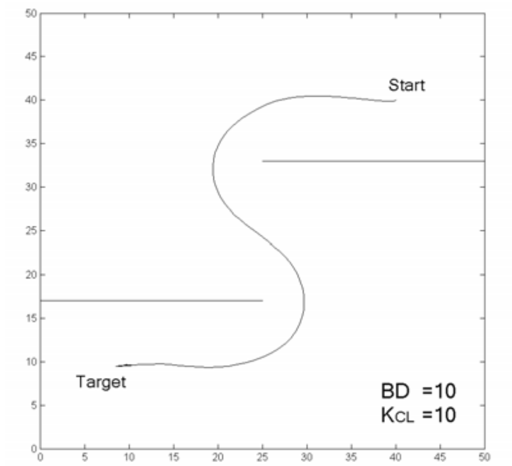

Figure-17: Trajectory, NADF and clamping.

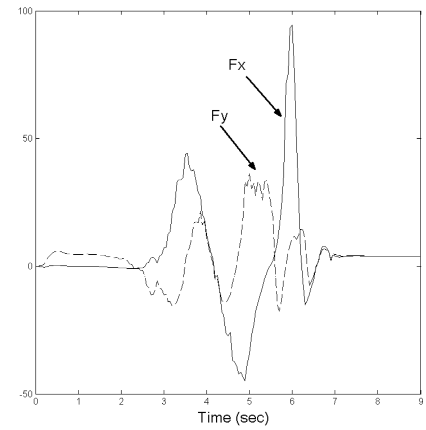

Figure-18: x and y control force components - external force present.

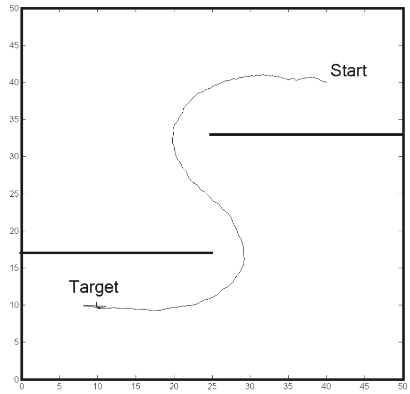

Figure-19: Trajectory - sliding mode control.

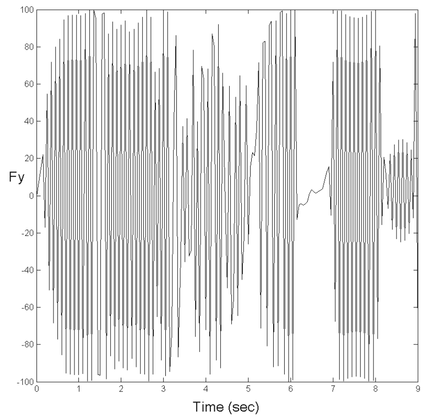

Figure-20- y force control component.

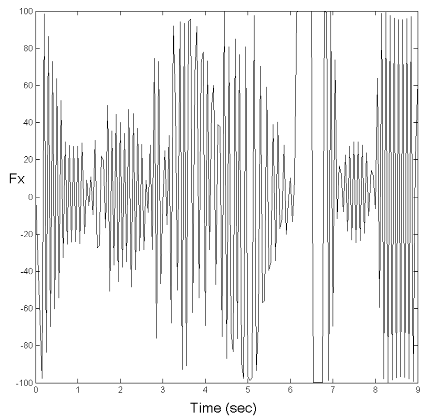

Figure-21: x force control component.

4. Iterative error removal:

The iterative procedure to remove the steady state error suggested in the previous section is tested using a simple pendulum with concentrated mass m=1Kg and length L=1m. The dynamic equation of the pendulum is:

$$m \cdot L \cdot \ddot{\Theta} + m \cdot \mathbf{g} \cdot \sin(\Theta) = u \qquad (71)$$

where g is the acceleration constant and u is the external applied control torque. A simple controller with position and velocity feedback (72) is used to move the pendulum from $\Theta=0$ to $\Theta=\pi/2$.

$$u = -k \cdot \Theta - b \cdot \dot{\Theta} \qquad (72)$$

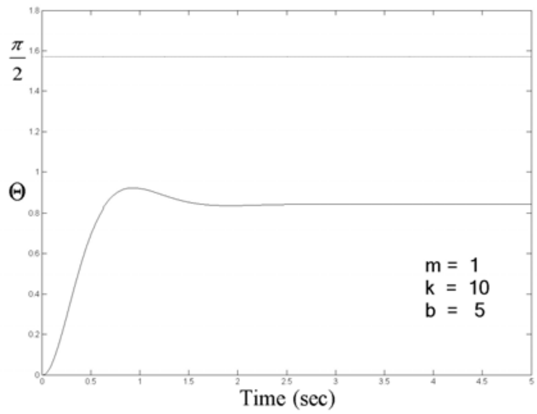

Figure-22: Steady state error caused by weight of pendulum.

As can be seen from figure-22, the weight of the pendulum causes significant steady state error. In order to remove the error, the switching circuit suggested in V.2 is added to the controller. Different switching thresholds are used to assess the sensitivity of the procedure to the presence of transients (figure-23). As can be seen, the error was eliminated in all cases. Although the iterative error cancellation procedure was designed to be used when transients fade away and motion settles, simulation shows that the procedure exhibits little sensitivity to the presence of transients that enables us to loosely choose the threshold α. Actually, the simulation reveals that better results in terms of having a lower settling time could be obtained if switching is carried out before motion completely settles. In figure-24 the effect of the forward gain on the speed of convergence is shown. As expected, the higher the forward gain is the faster the system converges to its target.

If the drift term cannot be represented as the gradient of a scalar function, the iterative procedure may still work. In equation (73) a random drift term, χ, is added to the system equation of the pendulum:

$$m \cdot L \cdot \ddot{\Theta} + \mathbf{g} \cdot \sin(\Theta) + 10 + 10 * \chi = u \qquad (73)$$

where χ is white noise uniformly distributed between -0.5 and 0.5. The iterative procedure is used with no modification to cancel this type of drift. As can be seen from figure-25, the procedure was able to converge in a statistical sense to the reference.

The iterative blind , error cancellation procedure was also simulated for a two-link, three degrees-of-freedom arm robot manipulator. The procedure was able to effectively remove the error in few iterations.

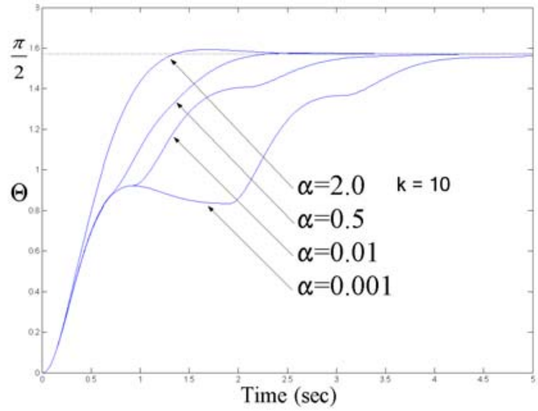

Figure-23: Error cancellation using switching circuit - different thresholds.

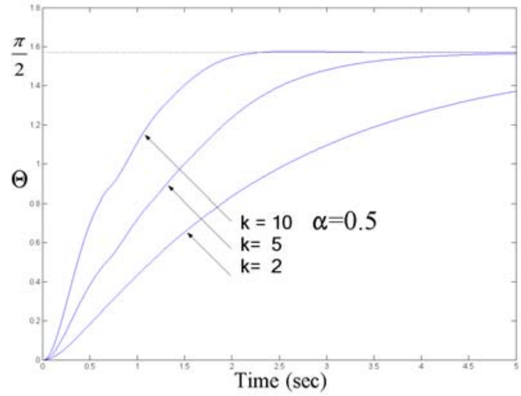

Figure-24: Effect of forward gain on convergence.

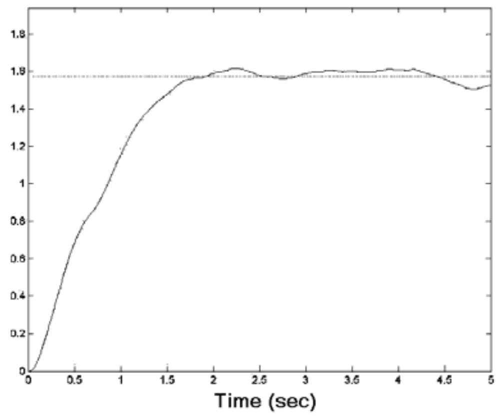

Figure-25: error cancellation - random drift term.

VII. Conclusions

In this paper the capabilities of the HPF approach are extended to tackle the kinodynamic planning case. The extension is provably-correct and bypasses many of the problems encountered by previous approaches. It is based on a novel type of nonlinear, passive damping forces called NADFs. The suggested approach enjoys several attractive properties. It is easy to tune; it can generate a well-behaved control signal; the approach is flexible and may be applied in a variety of situations, it is provably-correct; it is resistant to sensor noise; it does not require exact knowledge of system dynamics, and it can tackle dissipative systems as well as systems under the influence of external forces. The use of the NADF approach extends beyond a single dynamical agent. It can be adapted for use with a multi-robot dynamical system [30], as well as robots with nonholonomic constrains [31]. It ought to be emphasized that most of the problems attributed to the potential field approach, namely the narrow corridor effect, are a result of the misunderstanding of the dual role a potential field plays as a motion actuator and a guidance provider [33]. The NADF approach is a step forward in taking both of these roles into account.

Appendix-1

A. *Definition*: Let V(x) be a smooth (at least twice differentiable) scalar function ($V(x): R^N \to R$). A point xo is called a critical point of V if the gradient vanishes at that point ($\nabla V(xo)=0$); otherwise, xo is regular. A critical point is Morse, if its Hessian matrix (H(xo)) is nonsingular. V(x) is Morse if all of its critical points are Morse [24].

B. *Proposition*: If V(x) is a harmonic function defined in an N-dimensional space (R^N) on an open set Ω, then the Hessian matrix at every critical point of V is nonsingular, i.e. V is Morse.

*Proof***:** There are two properties of harmonic functions that are used in the proof:

1- a harmonic function (V(x)) defined on an open set Ω contains no maxima or minima, local or global in Ω. An extrema of V(x) can only occur at the boundary of Ω,

2- if V(x) is constant in any open subset of Ω, then it is constant for all Ω. Other properties of harmonic functions may be found in [26].

Let xo be a critical point of V(x) inside Ω. Since no maxima or minima of V exist inside Ω, xo has to be a saddle point. Let V(x) be represented in the neighborhood of xo using a second order Taylor series expansion:

$$V(x) = V(xo) + \nabla V(xo)^T (x - xo) + \frac{1}{2}(x - xo)^T H(xo)(x - xo) \qquad \|x\text{-}xo\| << 1. \tag{74}$$

Since Xo is a critical point of V, we have:

$$V' = V(x) - V(xo) = \frac{1}{2}(x - xo)^T H(xo)(x - xo) \qquad \|x\text{-}xo\| << 1. \tag{75}$$

Notice that adding or subtracting a constant from a harmonic function yields another harmonic function , i.e. V` is also harmonic. Using eigenvalue decomposition [25]:

$$V' = \frac{1}{2}(x - xo)^T U^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} U(x - xo)$$

$$= \frac{1}{2}\xi^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} \xi \quad = \quad \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{i=1}}^{\mathbf{N}} \lambda_{\mathbf{i}} \xi_{\mathbf{i}}^2 \tag{76}$$

where U is an orthonormal matrix of eigenvectors, λ's are the eigenvalues of H(xo), and $\xi=[\xi_1\ \xi_2 \,..\xi_N]^T = U(x\text{-}xo)$. Since V` is harmonic, it cannot be zero on any open subset Ω; otherwise, it will be zero for all Ω, which is not the case. This can only be true if and only if all the λ_i's are nonzero. In other words, the Hessian of V at a critical point xo is nonsingular. This makes the harmonic function V also a Morse function.

Appendix-2

Constructing an NADF force requires that the component of motion normal to -∇V be computed. Explicit computation of such component requires that N-1 set of basis vectors fully spanning the normal space be constructed. Although explicitly constructing such basis in R^N is possible, it is desirable that the normal component of motion be computed using an indirect approach that relies only on -∇V. This may be carried-out using the following steps:

1- compute the component of motion in-phase with -∇V (x_r),

$$x_{\mathbf{r}} = \dot{x}^{\mathbf{T}} \frac{-\nabla V(x)}{\|\nabla V(x)\|}, \tag{77}$$

2- remove the in-phase component from $\dot{X}$ creating the vector,

$$x_n = \dot{x} - x_{\mathbf{r}} \cdot \frac{-\nabla V(x)}{\|\nabla V(x)\|}, \tag{78}$$

3- normalize $\mathbf{x}_n$ to obtain the normal vector $\boldsymbol{\mu}$,:

$$\mu = \frac{x_n}{\|x_n\|}, \tag{79}$$

4- The orthogonal component may now be computed as:

$$\left|\dot{x}^{\mathbf{T}}\mu\right|\mu. \tag{80}$$

The following example demonstrates that the above process is equivalent to the direct procedure.

Example: at a certain point in space let $-\nabla V=[1/\sqrt{2} \quad 1/\sqrt{2}]^T$, $\mathbf{n} = [-1/\sqrt{2} \quad 1/\sqrt{2}]^T$ and $\dot{x} = [0.6\ -1]^T$. Using the direct procedure the normal component is:

$$(\dot{x}^T \mathbf{n})\cdot\mathbf{n} = [0.2 \ \ -0.2]^T . \tag{81}$$

Using the indirect procedure, we have:

$$x_r = \dot{x}^T (\mathbf{-\nabla V}) = -0.28284, \tag{82}$$

$$x_n = \dot{x} - x_r\cdot(\mathbf{-\nabla V}) = [0.8 \ \ -0.8]^T , \tag{83}$$

normalizing $\mathbf{x}_n$ we have:

$$\boldsymbol{\mu} = [1/\sqrt{2} \quad -1/\sqrt{2}]^T . \tag{84}$$

The orthogonal component of motion is:

$$|\dot{x}^T \boldsymbol{\mu}|\cdot\boldsymbol{\mu} = [0.2 \ \ -0.2]^T . \tag{85}$$

As can be seen, the answer is the same as the one from the direct approach.

Acknowledgment: The author would like to thank Dr. Tareq AL-Naffouri for his help in proving proposition-5. Also thanks go to KFUPM for its support.

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