Chapter 6

Digital Filters: Analysis, Design and Structures

6.1 Introduction

LTI system modifies input.

Filter - LTI system that modifies the input such that it removes certain undesired components

Undesired components - frequency components, filter becomes frequency selective.

LTI system performs a type of discrimination among the various frequency components, hence, LTI and filter are *synonymous*.

Simple filters - apply brute force method.

Complicated filters - sophisticated techniques

Filtering operation - frequency response characteristics of the LTI, namely, $H(\omega)$.

$H(\omega)$ depends on the choice of the parameters, i.e, the coefficients, a_k and b_k of the DE governing the system.

LTI - output given by:

$$Y(\omega) = X(\omega)H(\omega) \qquad 6.1.1$$

$H(\omega)$ - *weighting function* or *spectral shaping function* to different frequency components in the input signal (frequency shaping filter)

6.2 Ideal Filter Characteristics

Classification of Filters (according to their frequency domain characteristics)
LPF, HPF, BPF, BRF

Fig. 6.2.1

6.2.1 Features

1. Constant gain in the PB and zero gain in SB.

2. Linear Phase Response.

Flat frequency response, large phase shifts in the PB, distortion in its waveform.

Filter - phase shift varies linearly with frequency, equivalent to *constant delay* for signal in PB. All frequency components will be shifted by same delay, no distortion in the waveform.

Sharp transitions between pass and stop bands not physically realizable.

Causality and its implications

The FR of an ideal LPF is given below:

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < \omega < \pi \end{cases} \qquad 6.5.1$$

The IR of this filter is

$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c \sin \omega_c n}{\pi \omega_c n} & n \neq 0 \end{cases} \qquad 6.5.2$$

The plot shows *h(n)* when $\omega_c = \pi/4$. Fig. 6.5.1

Ideal LPF is non causal and hence, not practically realizable.

One solution: make $h(n) = 0$ for $n < n_0$, that is, introduce a large delay.

Results in non ideal FR and setting $h(n) = 0$ for $n < n_0$, FS expansion results in Gibbs oscillations.

Necessary & sufficient condition for causality - ***Paley Wiener Theorem.***

To obtain practical filters, relax our requirements on the PB and SB by permitting deviations from the ideal response.

6.5.2 Characteristics of Practical Filters

Ideal FR, desirable but not necessary in most of the applications.

Tolerable to have small variations, ripples, in the PB and small nonzero values in the SB.

Transition of the FR from the PB to SB - ***transition band.***

Band edge frequency, ω_p - edge of the PB

ω_s - beginning of the SB

ω_s - ω_p - transition band.

Width of the PB - ***bandwidth*** of the filter, ω_p.

Fig 6.5.1

Ripple in PB - δ_1, in log scale the ripple in PB is $20 \log \delta_1$ *db*

$|H(\omega)|$ - $1 \pm \delta_1$.

SB ripple - δ_2, in log scale $20 \log \delta_2$.

In filter design, one can specify the maximum tolerable δ_1 and δ_2, ω_s, and ω_p.

Select a_k and b_k which best approximates the desired specifications.

Concerned with design of only LPF

Other types of filters, convert LPF appropriate Frequency transformations.

6. 3 Frequency Transformations

Suppose CT LPF transfer function is H(s) with a pass band edge frequency Ω_p.

Convert it another LPF with a PB edge, $\Omega_p^{`}$ required transformation is

$$s \to \frac{\Omega_p}{\Omega_p^{'}} s$$

Hence the new LPF is

$$H_l(s) = H_p\left[\left(\frac{\Omega_p}{\Omega_p^{'}}\right)s\right]$$

$H_p(s)$ is the prototype LPF

To convert LPF $\to$ HPF with PS edge Ω_p'

$$s \to \frac{\Omega_p \Omega_p^{'}}{s}$$

System function of the HPF is

$$H_h(s) = H_p\left(\frac{\Omega_p \Omega_p^{'}}{s}\right)$$

Convert LPF with PB edge Ω_p to BPF, perform

$$s \to \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

New system function is

$$H_b(s) = H_p\left(\Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}\right)$$

Do a similar transformation to convert an LPF to BSF.

$$s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

$$H_{bs}(s) = H_p\left(\Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}\right)$$

Table 8.12

Mappings - non linear, may distort the characteristics of the resulting filters, amplitudes responses are preserved.

6.4 Design of Analog Filters

6.4.1 Butterworth Filter

LP BF are all pole filters characterized by the magnitude function

$$\left| H(\Omega) = \frac{1}{(1+\Omega)^{2N}} \right|$$

N is the order of the filter.

Magnitude - monotically decreasing function with Ω, with unity at $\Omega=0$ and for $\Omega=1$, the magnitude is $1/2^{1/2}$ for all N.

Fig. shows magnitude characteristics for different N, N is increased the characteristic approaches ideal filter.

Called a maximally flat filter, for a given N, the maximum number of derivatives of the magnitude function is zero at the origin.

Filter transfer function:

$$H(s)H(-s)|_{s=j\Omega} = |H(\omega)|^2$$

$$= \frac{1}{1+\left[\frac{(j\Omega)^2}{j^2}\right]^N}$$

$$= \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

Poles

$$\left(\frac{s}{j}\right)^{2N} = -1$$

$$= \exp[\mathrm{j}(2\mathrm{k}-1)\pi], \qquad k = 0,1,.. \quad ,2N-1$$

$$s_k = \exp\left[j(2k+N-1)\frac{\pi}{2N}\right], \qquad k = 0,1,. \;.. \quad 2N-1$$

Substituting

$s_k = \sigma_k + j\lambda_k$, separate into real and imaginary parts as:

$$\sigma_k = \cos\left(\frac{[2k+N-1]}{2N}\pi\right) = \sin\left(\frac{[2k-1]\pi}{2N}\right)$$

$$\lambda_k = \sin\left(\frac{[2k+N-1]}{2N}\pi\right) = \cos\left(\frac{[2k-1]\pi}{2N}\right)$$

2N roots - spaced around unit circle at intervals of *π/2N*, no roots on the ***j*** axis, *N* roots on either side of the ***j*** axis.

The poles & zeros of *H(s)* - mirror images of the poles and zeros of *H(-s).*

For stable filter - associate the roots in the LH plane.

Considered magnitude as 1, that is normalized cut off. In practice, multiply

Example N=3

Table 1 lists the *H(s)* in the factored form from *N=1* to *N=8.*

Table 2 gives the coefficients of the polynomial obtained by multiplying all the factors.

To get a filter with a 3dB cutoff at Ω_c,

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Consider LP BF satisfying the following conditions

$$\begin{aligned} H(\Omega) &\geq 1-\delta_1, \qquad & |\Omega| \leq \Omega_p \\ &\leq \delta_2, \qquad & |\Omega| > \Omega_s \end{aligned}$$

Need to determine *N* and Ω_c. Choose

$$|H(\Omega_p)| = 1-\delta_1, \text{ the ratio to be used is } \frac{\Omega_p}{\Omega_c}, \qquad 6.4.1$$

$$|H(\Omega_s)| = \delta_2, \text{ the ratio to be used is } \frac{\Omega_s}{\Omega_c}, \qquad 6.4.2$$

Substituting in the original equation and simplifying

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \left(\frac{1}{1-\delta_1}\right)^2 - 1$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \left(\frac{1}{\delta_2}\right)^2 - 1$$

Eliminating Ω_c and solving for *N*

$$N = \frac{1}{2}\left[\frac{\log \frac{\delta_1 (2-\delta)\delta_2^2}{(1-\delta_1)^2 (1-\delta_2^2)}}{\log \frac{\Omega_p}{\Omega_s}}\right]$$

Round up the value obtained for *N*.

This can be used in Eqn 6.4.1 or 6.4.2 to obtain Ω_c.

Steps
1. Determine *N* and round it up
2. Determine Ω_c from either of the equations
3. From N calculated determine the denominator polynomial from the tables

4. Determine the unnormalized transfer function by replacing s in *H(s)* found in 3 by s/Ω_c. The filter will then have a unity gain at DC.

6.4.2 The Chebyshev Filter

BF approximates ideal LPF for values of Ω near *0,* but the fall off rate with in the transition band is low.

CF has a sharper cut off rate in the transition band, however, it has ripples in the PB (I) and SB(II). For filter of same order, the CF has a smaller TB.

CF approximation is complex

Present the steps needed to determine *H(s).*

The CF is based on Chebyshev polynomials, defined as:

$$C_N(\omega) = \cos(N\cos^{-1}\Omega), \qquad |\Omega| \le 1$$
$$= \cosh(N\cosh^{-1}\Omega), \qquad |\Omega| > 1$$

Also defined by the recursion formula

$$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) - C_{N-2}(\Omega), \qquad C_0(\Omega) = 1 \text{ and } C_1(\Omega) = \Omega$$

Example

The Chebyshev LP characteristic of order N is defined in terms of $C_N(\Omega)$ by

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\Omega)}$$

1. Note the following properties:

$|C_N(\Omega)| \le 1$ for all $|\Omega| \le 1$

$C_N(1) = 1$ for all N

All roots of the polynomial lie in the interval $-1 \le \Omega \le 1$

$|H(\Omega)|^2$ oscillates about unity such that the maximum value is 1 and the minimum value is $1/(1+\varepsilon^2)$.

As Ω becomes large, $C_N(\Omega)$ becomes large and $|H(\Omega)|$ approaches 0 rapidly providing an approximation to the ideal LPF.

Fig shows the magnitude characteristics
For N odd, $C_N^2(0) = 0$, $|H(\Omega)|^2=1$
For n even , $C_N^2(0) = 1$, $|H(\Omega)|^2=1/(1+\varepsilon^2)$.

And

$$H(1) = \frac{1}{\sqrt{1+\varepsilon^2}}.$$

For large values of Ω, that is in the SB, we can approximate

$$H(\Omega) = \frac{1}{\varepsilon C_N(\Omega)}$$

The attenuation from $\Omega = 0$ is:
Loss $= -20\log 10|H(\Omega)|$
$= 20 \log \varepsilon + 20 \log C_N(\Omega)$

For large Ω, $C_N(\Omega)$ can be approximated by $2^{N-1}\Omega^N$
So that the loss can be written as
Loss $= 20 \log \varepsilon + 6(N-1) + 20N \log(\Omega)$
N and ε can be determined using the specifications from the above equations.

The poles of H(s) are given as

$$\sigma_k = \sin\left(\frac{[2k-1]\pi}{2N}\right)\sinh\beta$$

$$\lambda_k = \cos\left(\frac{[2k-1]\pi}{2N}\right)\cosh\beta$$

Where $\beta = \frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}$

It follows that the poles are located on an ellipse in the s plane

$$\frac{\sigma_k^2}{\sinh^2\beta} + \frac{\lambda_k^2}{\cosh^2\beta}$$

We also see that the poles of CF are related to the poles f the BF.
Fig 10.3.6

6.5 Design of Digital Filters

Desired characteristics - specified in FD (magnitude and phase response).

Design process - determine coefficients of a causal FIR or IIR that closely approximates the desired specifications.

The desired characteristics determine whether FIR or IIR is required.

If ***linear phase*** - FIR is chosen.

IIRs have lower side lobes in the SB than FIR for the same number of parameters.

If distortion is tolerable, IIR is preferable since it uses fewer parameters and hence, lower storage and less computational complexity.

6.6 Design of Simple Digital Filters by pole zero placement.

Principle - place *poles* near points of the unit circle corresponding to frequencies to be emphasized and to place *zeros* near frequencies that are to be de-emphasized.

The following constraints apply.
1. All poles- placed *with in the unit circle*. Zeros - placed *anywhere.*

2. Complex zeros and poles must occur in conjugate pairs for the coefficients to be real.

For a given pole zero pattern, *H(z)* can be written as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)} \qquad 6.6.1$$

b_0 is a gain constant selected to normalize the FR at some specified frequency, that is, $|H(\omega_0)|=1$

ω_0 is the PB frequency.

$N \geq M$, so that the filter has nontrivial poles than zeros.

6.6.1 <u>Design of LPF, HPF and BPF</u>

6.6.1.1 Low Pass Filters (LPF)
Poles - placed near the unit circle at points corresponding to low frequencies, ie., $\boldsymbol{\omega = 0}$ and zeros should be placed near or on the unit circle at points corresponding to $\boldsymbol{\omega = \pi}$.

It is the other way for HPF.

Fig illustrates this.

Magnitude and phase response of a single pole filter is shown;

$$H_1(z) = \frac{1-a}{1-az^{-1}} \qquad 6.6.2$$

The magnitude and the phase response is shown for *a = 0.9*

Addition of a zero at z = -1, further attenuates the response at high frequencies as shown by the following:

$$H_2(z) = \frac{1-a}{1-az^{-1}} \frac{1+z^{-1}}{2} \qquad 6.6.3$$

HPF can be obtained by the reflecting the pole zero about the imaginary axis.

$$H_3(z) = \frac{1-a}{1+az^{-1}} \frac{1-z^{-1}}{2} \qquad 6.6.4$$

Fig.

Same principles for BPF.

BPF - contain one or more pairs of complex conjugate poles near the unit circle, near the frequency band of interest.

The above method provides an insight into the effect that poles and zeros have on FR characteristics of the system and is not a good design method.

6.7 FIR Filters

FIRs - no counter parts in the analog domain but do have linear phase characteristics

6.7.1 Linear Phase Filters

Transfer function of an FIR causal filter :

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

h(n) is the impulse response of the filter. The DE governing the FIR is obtained by taking the IZT

$$y(iT) = \sum_{n=0}^{N-1} h(n) x(iT - nT)$$

Convolution sum.

FT of the finite sequence h(n) is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{j\omega T} = |H(\omega)| e^{j\theta(\omega)}$$

The magnitude and the phase are defined by the following:

$$M(\omega) = |H(\omega)|$$

$$\theta(\omega) = \tan^{-1}\left(\frac{-\operatorname{Im} H(\omega)}{\operatorname{Re} H(\omega)} \right)$$

Define the phase delay and the group delay as follows:

$$\tau_p = -\frac{\theta(\omega)}{\omega}$$

and

$$\tau_g = -\frac{d\theta}{d\omega}$$

Derivative of phase with respect to frequency has the units of delay.

This is the time delay a frequency component undergoes as it passes from the input to the output.

When θ is linear, the group delay is a constant, that is all frequency components of the input signal undergo the same delay, called Linear Phase or constant delay filters.

phase response to be linear, we require

$$\theta(\omega) = -\tau\omega$$

$$\theta(\omega) = -\tau\omega = \tan^{-1} \frac{-\sum_{n=0}^{N-1} h(n) \sin \omega nT}{\sum_{n=0}^{N-1} h(n) \cos \omega nT}$$

or

$$\tan \omega\tau = \frac{\sum_{n=0}^{N-1} h(n) \sin \omega nT}{\sum_{n=0}^{N-1} h(n) \cos \omega nT}$$

Finally

$$\sum_{n=0}^{N-1} h(n)\sin(\omega\tau - \omega nT) = 0$$

The solution to this is given by the following:

$$\tau = \frac{(N-1)T}{2}$$
$$h(n) = h(N-1-n)$$

For the filter to have constant phase and group delays the above conditions have to be met.

Notice that the coefficients are mirror image of each other.

6.7.2 FIR Filter Design

IIR filters do not have, in general, linear phase characteristics.

FIR filter of length N has linear phase characteristics if the following is met

$$h(n) = h(N-1-n)$$

Consider N odd and even separately

For N even

$$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$
$$= \sum_{n=0}^{N/2-1} h(n)e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)e^{-j\omega n}$$

replacing *n* by *N-n-1*, we have

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)e^{-j\omega n} + \sum_{n=0}^{N/2-1} h(n)e^{-j\omega(N-n-1)}$$

which can be written as

$$H(\omega) = [\sum_{n=0}^{N/2-1} 2h(n)\cos\left[\omega\left(n - \frac{N-1}{2}\right)\right]]\exp(-j\omega\left(\frac{N-1}{2}\right))$$

Similarly for *N* odd

$$H(\omega) = [h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{N/2-1} 2h(n)\cos\left[\omega\left(n - \frac{N-1}{2}\right)\right]]\exp(-j\omega\left(\frac{N-1}{2}\right))$$

Terms in the square brackets are real, so that the phase is given by the complex exponential:

$$\theta(\omega) = \omega\left(\frac{N-1}{2}\right),$$

compare this with

$$\theta(\omega) = -\tau\omega$$

System has a linear phase shift with a corresponding delay of (N-1)/2 samples.

If $H_d(\omega)$ is the desired FR of say an ideal LPF, which is symmetric about the origin, the corresponding impulse response, $h_d(n)$ is symmetric about 0, but in general is of infinite duration as shown:

$$\begin{aligned} H(\omega) &= 1, \quad |\omega| \le \omega_c \\ &= 0, \quad \omega_c < \omega \le \pi \end{aligned}$$

The impulse response is:

$$h_d(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin\omega_c n}{\omega_c n} & n \neq 0 \end{cases}$$

A plot is shown in Fig 8.1.

Ideal filter is non causal and hence cannot be realized in practice.

One way of obtaining an FIR of length N, truncate the infinite sequence for $h_d(n)$ by multiplying by a window, $w(n)$.

$$w(n) = \begin{cases} 1 & n = 0,1....M-1 \\ 0 & otherwise \end{cases}$$

The unit sample response is then:

$$h(n) = h_d(n)w(n)$$
$$= \begin{cases} h_d & n = 0,1....M-1 \\ 0 & otherwise \end{cases}$$

Consider the effect of windowing.

Multiplication in the TD is equivalent to convolution in the FD. Hence

$h_d(n)w(n)$ in the TD is $H_d(\omega) * W(\omega)$ in the FD.

where

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n}$$

The FR, H(ω) of the truncated filter is

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\upsilon)W(\omega-\upsilon)d\upsilon$$

The FT of the rectangular window is:

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = e^{-j\omega(M-1)/2}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

Magnitude response is shown in Fig for M=31

The width of the ML = 4π/M.

M increases, the width of the ML decreases.

SLs are high and remain unaffected as M increases

Height of SL increases with increasing M in a way such that the area under the SL remain the same.

The FR of the FIR is determined by the characteristics of the window:
1. Smoothes $H_d(\omega)$

2.As M is increased, the ML becomes narrower, hence the smoothing is reduced.
SLs increase and cause ringing.

Steps in designing FIR

1. From the desired $H_d(\omega)$, find $h_d(n)$
2. Multiply $h_d(n)$ by $w(n)$
3. Determine $H(z)$.

Lets look at the system function of the FIR

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$$

Substitute z^{-1} for z in the above and multiply both the sides with $z^{-(M-1)}$, we have

$$z^{-(M-1)} H(z^{-1}) = \sum_{k=0}^{M-1} h(k) z^{k} z^{-(M-1)}$$

$$= \sum_{k=0}^{M-1} h(k) z^{-(M-1-k)}$$

Use the fact $h(k) = \pm h(M-1-k)$ and substitute $l=(M-1-k)$

$$z^{-(M-1)} H(z^{-1}) = \sum_{l=0}^{M-1} h(M-1-l) z^{-l}$$

$$= \pm H(z)$$

Implies that the roots of $H(z)$ are identical to the roots of $H(z^{-1})$.

This means that the roots of $H(z)$ must occur in reciprocal pairs, if z_1 is a root then $1/z_1$ is also a root.

If $h(n)$ is real then complex valued roots must occur in conjugate pairs.
If z_1 is a complex root, the z_1^* is a complex conjugate. $1/z_1^*$ is also a zero.

Fig shows the symmetry of the zeros.

Window Functions

Hanning, Hamming, Blackman windows have lower side lobes.

For the same M, the width of the main lobe is more and hence more of smearing effect and more smoothing.

transition width also increases. To reduce the transition width, increase M and this results in a larger filter.

Types of windows: Figs from 8.6

IIR Filters

Well established techniques available for the design of IIR filters.

Two most commonly used methods based on analog filter design techniques and we confine our discussions to the design of the LPF.

Procedure
Convert given digital filter specifications to equivalent analog filter specifications

Design an analog filter transfer function $H_a(s)$

Convert it into an equivalent DT transfer function.

The analog filter can either be described by its system function

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

where α_k and β_k are the filter coefficients or by its impulse response

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

or by linear constant coefficient differential equation

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x}{dt^k}$$

Each representation allows a method for converting into digital domain.

For *H(s)* to be stable - poles should lie in the LH of the s plane.

Hence for the digital filter to be stable, the conversion should result in

1. $j\Omega$ axis should map into a unit circle in the z plane.
2. The LH of *s* plane should map into the unit circle in the *z* plane.

For a filter to be linear phase we have derived the condition that

$$H(z) = \pm z^{-N} (z^{-1})$$

If this were the case, the filter would have mirror image pole outside the unit circle for every pole that is inside which means that the filter would be unstable.

Hence, a causal and stable IIR filter cannot have linear phase.

In the design, specify the magnitude response and accept whatever phase response is obtained from the design method.

6.7.3 IIR Filter Design by Impulse Invariance

Objective - design an IIR filter having a Unit Sample Response that is the sampled version of the impulse response of an analog filter.

Recall - when a CT signal $x_a(t)$ with FT $X_a(F)$ is sampled at $F_s=1/T$, the FT of the sampled version of $x_a(t)$ is periodic repetition of a scaled spectrum, $F_s\, X_a(F)$ with a period F_s. In mathematical terms

$$X(F/F_s) = F_s \sum_{k=-\infty}^{\infty} X_a\left[F - kF_s\right]$$

with $f = \dfrac{F}{F_s}$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a\left[(f-k)F_s\right]$$

f is the normalized frequency.

Aliasing, if F_s is < twice the highest frequency in $X_a(F)$.

In our case, it is the impulse response $h(t)$ with a FT $H_a(F)$ that is being sampled. Hence the digital filter has the frequency response

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a\left[(f-k)F_s\right]$$

or

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a\left[(\omega - 2\pi k)F_s\right]$$

Or

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left[(\Omega - \frac{2\pi k}{T})\right]$$

Fig 8.31 (proakis) shows both the responses

If F_s is sufficiently large, aliasing could be avoided.

It is for this reason that this method cannot be used for designing HPF

Mapping between the s and the z planes, consider s=jΩ in the following general expression.

Same as the above with j supressed.

$$H(z)\,|_{z=e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left[(s - j\frac{2\pi k}{T})\right]$$

Consider the mapping implied by $z=e^{sT}$

$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$

so

$r = e^{\sigma T}$ and

$\omega = \Omega T$

We have used $s = \sigma + j\Omega$.

Mapping is such the left half $(\sigma \le 0)$ of s plane maps into unit circle and RH of s plane maps into outside of the unit circle.

$j\Omega$ axis maps into the unit circle, mapping is not one to one.
ω is unique in $-\pi, \pi$. The mapping $\omega = \Omega T$ implies that the interval
$-\pi/T \le \Omega \le \pi/T$ maps in to $-\pi \le \omega \le \pi$.

LP maps into unit circle, in general $-(2k-1)\ \pi/T \le \Omega \le \pi/T(2k+1)$ is true.

Mapping is many to one and this due to aliasing.
Fig.

Look at the how the impulse invariance method affects the characteristics of the resulting filter, look at the poles.

Express the system function of the analog filter as;

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{(s - p_i)}$$

The TD expression is the ILT

$$h_a(t) = L^{-1}\{H_a(s)\} = \sum_{i=1}^{M} A_i e^{pit}$$

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{pitN}$$

$$H(z) = Z\{h(n)\} = \sum_{i=1}^{M} \frac{A_i z}{z - e^{Tpi}}$$

Zeros do not satisfy the above relationship.

could be generalized this for multiple order poles.

Steps:
1. Determine the equivalent analog frequencies from the digital PB and cut off frequencies: $\omega_p \rightarrow \Omega_p$ and $\omega_s \rightarrow \Omega_s$
2. Obtain the analog Transfer function, $H_a(s)$ (BF and CF)
3. Expand $H_a(s)$ into partial fractions, determine ZT of each term and then combine terms to obtain the overall ZT.

Bilinear Transformation.
To avoid aliasing, need a special type of mapping.

Mapping- z domain unit circle into the vertical axis in the s plane, the interior of the unit circle into the LH plane in the s plane and the exterior of the unit circle into the open RH half in the s plane (conformal mapping).

The BT achieves this

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

To investigate this mapping substitute

$$z = re^{j\omega},\, s = \sigma + j\Omega$$

$$s = \frac{2}{T}\frac{re^{j\omega}-1}{re^{j\omega}+1}$$

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

equating the real and the imaginary parts

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]$$

$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

If r < 1, σ <0 and if r > 1, σ > 0.

Interior of unit circle is mapped into LH and the exterior is mapped to the RH of splane. Also, if r =1, σ = 0. Also

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\tan\frac{\omega}{2}$$

or

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

The relation between the frequency variables is shown in Fig

Observe entire Ω is mapped only once in $-\pi \le \omega \le \pi$.

Mapping is non linear and there is frequency warping