Problem 8-1

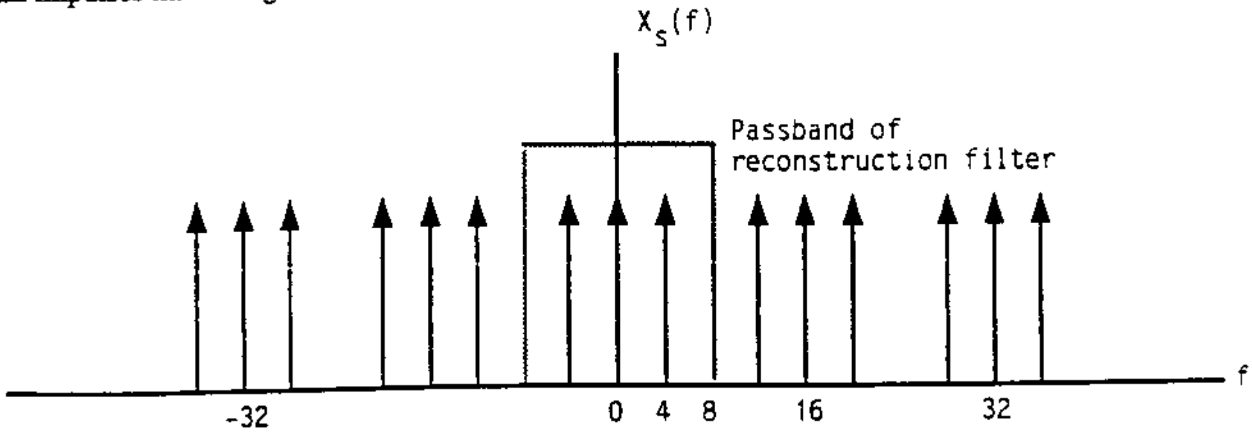
With

$$x(t) = 4 + 8\cos 8\pi t$$

and

$$X(f) = 4\delta(f) + 4\delta(f-4) + 4\delta(f+4)$$

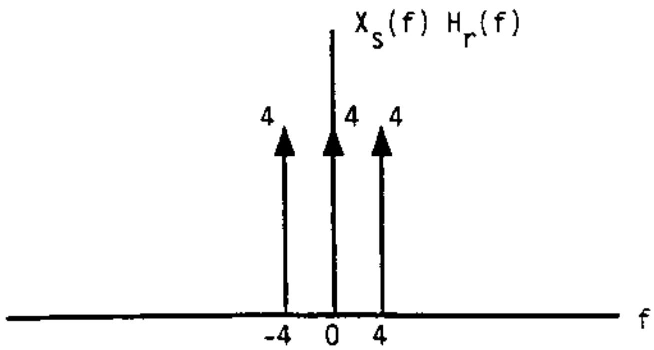
the specturm of the sampled signal appears as shown below. For a sampling frequency of 16 Hz all impulses have wieght 64.



The reconstruction filter has the transfer function

$$H_r(f) = \begin{cases} \frac{1}{16}, & |f| \le 8 \\ 0, & \text{otherwise} \end{cases}$$

The spectrum at the output of the reconstruction filter is



Clearly

$$X_s(f)\, H_r(f) = X(f)$$

which corresponds exactly to the time domain signal $x(t)$.

Problem 8-3

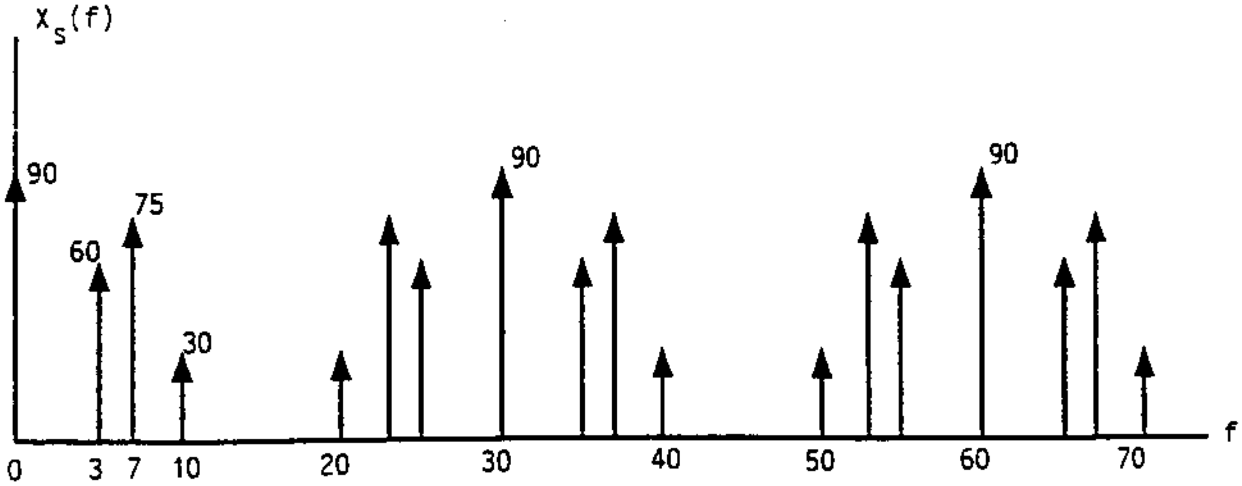
For

$$x(t) = 3 + 4\cos 10\pi t + 5\cos 14\pi t + 2\cos 20\pi t$$

we have

$$\begin{aligned} X(f) &= 3\delta(f) + 2\delta(f+5) + 2\delta(f-5) \\ &+ \frac{5}{2}\delta(f+7) + \frac{5}{2}\delta(f-7) \\ &+ \delta(f+10) + \delta(f-10) \end{aligned}$$

The dc component and the positive-frequency portion of the spectrum of the sampled signal is shown below for $f \leq 70$.



The original signal, $x(t)$, can be reconstructed from the sampled signal, $x_s(t)$, using a lowpass filt having the transfer function

$$H_r(f = \begin{cases} \frac{1}{30}, & |f| < 15 \\ 0, & \text{otherwise} \end{cases}$$

Problem 8-9

Since

$$H(f)=\frac{1}{1+j\left(\frac{f}{f_3}\right)^2}$$

it follows that the amplitude response is

$$|H(f)|=\frac{1}{\sqrt{1+(f/f_3)^2}}$$

and the amplitude response, expressed in dB, is

$$20\log_{10}|H(f)|=-10\log_{10}\left(1+\left(\frac{f}{f_s}\right)^2\right)$$

(a) Using the given definitions of δ_1 and δ_2 gives

$$-10\log_{10}\left(1+\left(\frac{W}{f_3}\right)^2\right)=\delta_1$$

from which

$$\frac{W}{f_3}=\sqrt{10^{-0.1\delta_1}-1}$$

and

$$\frac{f_s-W}{f_3}=\sqrt{10^{-0.1\delta_2}-1}$$

Since

$$\frac{f_s-W}{f_3}=\frac{f_s}{f_3}-\frac{W}{f_3}=\frac{f_s}{W}\frac{W}{f_3}-\frac{W}{f_3}$$

Solving for f_s/W gives

$$\frac{f_s}{W}=\frac{\left(\frac{W}{f_3}\right)+\left(\frac{f_s-W}{f_3}\right)}{\left(\frac{W}{f_3}\right)}$$

Substituting for $\frac{(f_s-W)}{f_3}$ and $\frac{W}{f_3}$ yields

$$f_s=W\frac{\sqrt{10^{-0.1\delta_1}-1}+\sqrt{10^{-0.1\delta_2}-1}}{\sqrt{10^{-0.1\delta_1}-1}}$$

(b) For $\delta_1=-3dB$ and $\delta_2=-30dB$

$$\frac{f_s}{W}=\frac{\sqrt{10^{0.3}-1}+\sqrt{10^{3}-1}}{10^{0.3}-1}=32.68$$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1}-1}+\sqrt{10^{3}-1}}{\sqrt{10^{0.1}-1}} = 123.07$$

For $\delta_1 = -1dB$ and $\delta_2 = -40dB$

$$\frac{f_s}{W} = \frac{\sqrt{10^{0.1}-1}+\sqrt{10^{4}-1}}{\sqrt{10^{0.1}-1}} = 196.51$$

Note that reducing the attenuation at $f = W$ from *–3* to *–1dB* results in the necessity of increasing the sampling frequency by a factor of *4*, while increasing the attenuation at $f = f_s - W$ from –30 to –40dB requires the sampling frequency to be increased by 60%. Large values of f_s/W can be avoided by using a higher-order filter.

Problem 8-19

(a) The z-transform of $x(nT)=\left(\frac{1}{5}\right)^n u(n)$ is

$$X(z)=\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n z^{-n}=\sum_{n=0}^{\infty}\left(\frac{1}{5}z^{-1}\right)^n$$

which is

$$X(z)=\frac{1}{1-\frac{1}{5}z^{-1}},\quad |z|>\frac{1}{5}$$

(b) The z-transform of $x(nT)=\left(-\frac{1}{5}\right)^n u(n)$ is

$$X(z)=\sum_{n=0}^{\infty}\left(-\frac{1}{5}\right)^n z^{-n}=\sum_{n=0}^{\infty}\left(-\frac{1}{5}z^{-1}\right)^n$$

which is

$$X(z)=\frac{1}{1+\frac{1}{5}z^{-1}},\quad |z|>\frac{1}{5}$$

(c) The z-transform of $x(nT)=u(n)+\left(\frac{3}{4}\right)^n u(n-4)$ is

$$X(z)=\sum_{n=0}^{\infty}z^{-n}+\sum_{n=4}^{\infty}\left(\frac{3}{4}\right)^n z^{-n}$$

which can be written

$$X(z)=\sum_{n=0}^{\infty}\left(z^{-1}\right)^n+\sum_{n=4}^{\infty}\left(\frac{3}{4}z^{-1}\right)^n$$

With the change of index $k=n-4$ in the second sum, this becomes

$$X(z)=\frac{1}{1-z^{-1}}+\sum_{k=0}^{\infty}\left(\frac{3}{4}z^{-1}\right)^{k+4}$$

This gives

$$X(z)=\frac{1}{1-z^{-1}}+\frac{\left(\frac{3}{4}z^{-1}\right)^4}{1-\frac{3}{4}z^{-1}},\quad |z|>1$$

The region of convergence results by recognizing that the first term exists for $|z| > 1$ and the second term exists for $|z| > \frac{3}{4}$. The z-transform exists only when both terms are defined. This requires that $|z| > 1$.

(d) The z-transform of $x(nT) = 2u(n) - 2u(n-8)$ is

$$X(z) = \sum_{n=0}^{\infty} 2z^{-n} - \sum_{n=8}^{\infty} 2z^{-n}$$

Letting $k = n - 8$ in the second sum is

$$X(z) = \sum_{n=0}^{\infty} 2(z^{-1})^{n} - \sum_{k=0}^{\infty} 2(z^{-1})^{k+8}$$

This gives

$$X(z) = \frac{2}{1-z^{-1}} - \frac{2z^{-8}}{1-z^{-1}}$$

or

$$X(z) = \frac{2(1-z^{-8})}{1-z^{-1}}, \quad |z| \neq 0$$

The above expression gives $X(z)$ in closed form. The region of convergence is justified by recognizing that $X(z)$ can also be written in terms of the <u>finite</u> sum.

$$X(z) = 2 + 2z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-4} + 2z^{-5} + 2z^{-6} + 2z^{-7}$$

The terms in the above series of the form z^{-k} for $k > 0$ are clearly defined for all z except $z = 0$.

Problem 8-29

(a) The long division is shown below

$$
\begin{array}{r}
1+0.30z^{-1}-0.21z^{-2}+0.347z^{-3}+0.3171z^{-4} \\
1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3}\,\big)\,1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
1-0.30z^{-1}+0.30z^{-2}-0.50z^{-3} \\
\hline
0.30z^{-1}-0.30z^{-2}+0.50z^{-3} \\
0.30z^{-1}-0.09z^{-2}+0.09z^{-3}-0.15z^{-4} \\
\hline
-0.21z^{-2}+0.41z^{-3}+0.15z^{-4} \\
-0.21z^{-2}+0.063z^{-3}-0.063z^{-4} \\
\hline
0.347z^{-3}+0.213z^{-4} \\
0.347z^{-3}-0.1041z^{-4} \\
\hline
0.3171z^{-4}
\end{array}
$$

Therefore

$$x(0)=1$$
$$x(T)=0.3$$
$$x(2T)=-0.21$$
$$x(3T)=0.347$$
$$x(4T)=0.3171$$

(b) Since

$$X(z)=\frac{1-0.72^{-1}}{\left(1+0.5z^{-1}\right)^{2}}=\frac{1-0.72^{-1}}{1+z^{-1}+0.25z^{-2}}$$

we write the long division as

$$
\begin{array}{r}
1-1.70z^{-1}+1.45z^{-2}-1.025z^{-3}+0.6625z^{-4} \\
1+z^{-1}+0.25z^{-2}\,\big)\,1-0.70z^{-1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
1+z^{-1}+0.25z^{-2} \\
\hline
-1.70z^{-1}-0.25z^{-2} \\
-1.70z^{-1}-1.70z^{-2}-0.425z^{-3} \\
\hline
1.45z^{-2}+0.425z^{-3} \\
1.45z^{-2}+1.45z^{-3}+0.3625z^{-4} \\
\hline
-1.025z^{-3}-0.3625z^{-4} \\
-1.025z^{-3}-1.025z^{-4} \\
\hline
0.6625z^{-4}
\end{array}
$$

Thus

$$x(0) = 1$$
$$x(T) = -1.7$$
$$x(2T) = 1.45$$
$$x(3T) = -1.025$$
$$x(4T) = 0.6625$$

(c) Since

$$X(z) = \frac{z^{-1}\left(1+0.2z^{-1}\right)^2}{\left(1-0.4z^{-2}\right)^2} = \frac{z^{-1}+0.4z^{-2}+0.04z^{-3}}{1-0.8z^{-2}+0.16z^{-4}}$$

The long division is (only enough to give the required values are shown)

$$\begin{array}{r|l}
 & z^{-1}+0.40z^{-2}+0.84z^{-3}+0.32z^{-4} \\ \hline
1-0.80z^{-2}+0.16z^{-4} & z^{-1}+0.40z^{-2}+0.04z^{-3} \\
 & z^{-1}-0.80z^{-3} \\ \hline
 & 0.40z^{-2}+0.84z^{-3} \\
 & 0.40z^{-2}-0.32z^{-4} \\ \hline
 & 0.84z^{-3}+0.32z^{-4} \\
 & 0.84z^{-3} \\ \hline
 & 0.32z^{-4}
\end{array}$$

Thus

$$x(0) = 0$$
$$x(T) = 1$$
$$x(2T) = 0.4$$
$$x(3T) = 0.84$$
$$x(4T) = 0.32$$

Thus

$$
\begin{aligned}
x(0) &= 1 \\
x(T) &= -1.7 \\
x(2T) &= 1.45 \\
x(3T) &= -1.025 \\
x(4T) &= 0.6625
\end{aligned}
$$

(c) Since

$$
X(z) = \frac{z^{-1}\left(1+0.2z^{-1}\right)^2}{\left(1-0.4z^{-2}\right)^2} = \frac{z^{-1}+0.4z^{-2}+0.04z^{-3}}{1-0.8z^{-2}+0.16z^{-4}}
$$

The long division is (only enough to give the required values are shown)

$$
\begin{array}{r|l}
 & z^{-1}+0.40z^{-2}+0.84z^{-3}+0.32z^{-4} \\
\hline
1-0.80z^{-2}+0.16z^{-4} & z^{-1}+0.40z^{-2}+0.04z^{-3} \\
 & z^{-1}-0.80z^{-3} \\
\hline
 & \quad 0.40z^{-2}+0.84z^{-3} \\
 & \quad 0.40z^{-2}-0.32z^{-4} \\
\hline
 & \qquad 0.84z^{-3}+0.32z^{-4} \\
 & \qquad 0.84z^{-3} \\
\hline
 & \qquad 0.32z^{-4}
\end{array}
$$

Thus

$$
\begin{aligned}
x(0) &= 0 \\
x(T) &= 1 \\
x(2T) &= 0.4 \\
x(3T) &= 0.84 \\
x(4T) &= 0.32
\end{aligned}
$$

Problem 8-32

(a) Since

$$X(z) = \frac{2}{(1-z^{-1})(1+0.2z^{-1})} = \frac{2z^2}{(z-1)(z+0.2)}$$

The partial fraction expansion is

$$\frac{X(z)}{z} = \frac{2z}{(z-1)(z+0.2)} = \frac{A}{z-1} + \frac{B}{z+0.2}$$

The value of A is

$$A = \frac{2}{1.2} = \frac{20}{12} = \frac{5}{3}$$

and the value of B is

$$B = \frac{2(-0.2)}{-1.2} = \frac{1}{3}$$

Thus

$$X(z) = \frac{5}{3}\frac{1}{1-z^{-1}} + \frac{1}{3}\frac{1}{1+0.2z^{-1}}$$

so that

$$x(nT) = \left[\frac{5}{3} + \frac{1}{3}(-0.2)^n\right]u(n)$$

(b) Since

$$X(z) = \frac{2}{(1-z^{-1})(1+z^{-1})} = \frac{2z^2}{(z-1)(z+1)}$$

The partial fraction expansion is

$$\frac{X(z)}{z} = \frac{2z}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

where

$$A = 1, \quad B = \frac{-2}{-2} = 1$$

Thus

$$X(z)=\frac{1}{1-z^{-1}}+\frac{1}{1+z^{-1}}$$

and

$$x(nT)=\left[1+(-1)^n\right]u(n)$$

(c) Since

$$X(z)=\frac{1+0.3z^{-1}}{\left(1+0.2z^{-1}\right)\left(1-0.4z^{-1}\right)}=\frac{z(z+0.3)}{(z+0.2)(z-0.4)}$$

Thus

$$\frac{X(z)}{z}=\frac{z+0.3}{(z+0.2)(z-0.4)}=\frac{A}{z+0.2}+\frac{B}{z-0.4}$$

where

$$A=\frac{0.1}{-0.6}=-\frac{1}{6}$$

and

$$B=\frac{0.7}{0.6}=\frac{7}{6}$$

Therefore $X(z)$ is given by

$$X(z)=-\frac{1}{6}\frac{1}{1+0.2z^{-1}}+\frac{7}{6}\frac{1}{1-0.4z^{-1}}$$

and $x(nT)$ is given by

$$x(nT)=\left[-\frac{1}{6}(-0.2)^n+\frac{7}{6}(0.4)^n\right]u(n)$$

(d) Since

$$X(z)=\frac{1+0.3z^{-1}}{\left(1+0.2z^{-1}\right)\left(2-0.4z^{-1}\right)}=\frac{z(0.5z+0.15)}{(z+0.2)(z-0.2)}$$

$$\frac{X(z)}{z} = \frac{0.5z + 0.15}{(z+0.2)(z-0.2)} = \frac{A}{z+0.2} + \frac{B}{z-0.2}$$

The value of A is

$$A = \frac{0.5(-0.2) + 0.15}{-0.4} = -\frac{0.05}{0.40} = -\frac{1}{8}$$

and the value of B is

$$B = \frac{0.5(0.2) + 0.15}{0.4} = \frac{25}{40} = \frac{5}{8}$$

This gives

$$X(z) = -\frac{1}{8}\frac{1}{1+0.2z^{-1}} + \frac{5}{8}\frac{1}{1-0.2z^{-1}}$$

Thus $x(nT)$ is given by

$$x(nT) = \left[-\frac{1}{8}(-0.2)^n + \frac{5}{8}(0.2)^n\right]u(n)$$