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EE 207 | Homework # 4 Solutions | October 18,1998

Problem 4-1

(a) The Fourier transform of this signal is

$$X_a(f) = \int_{-\infty}^{\infty} A\, e^{-\alpha t} u(t) e^{-j2\pi f t} dt$$

$$= \int_{0}^{\infty} A\, e^{-(\alpha + j2\pi f)t} dt$$

$$= A \frac{e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)} \Big|_0^\infty$$

$$= \frac{A}{\alpha + j2\pi f}$$

where the evaluation of the upper limit gives zero because it is given that $\alpha > 0$.
(b) The evaluation of this Fourier transform is similar to the one above:

$$X_a(f) = \int_{-\infty}^{\infty} A\, e^{\alpha t} u(-t) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{0} A\, e^{(\alpha - j2\pi f)t} dt$$

$$= A \frac{e^{-(\alpha + j2\pi f)t}}{(\alpha - j2\pi f)} \Big|_{-\infty}^{0}$$

$$= \frac{A}{\alpha - j2\pi f}$$

(c) This Fourier transform can be evaluated without integration by noting that the signal is the sum of the signals given in parts (a) and (b) and using superposition:

$$x_c(t) = Ae^{\alpha t}u(-t) + Ae^{-\alpha t}u(t) = x_a(t) + x_b(t)$$

Thus, by superposition,

$$X_c(f) = X_a(f) + X_b(f)$$

$$= A\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{\alpha - j2\pi f}\right]$$

$$= \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$$

(d) This signal is the difference of the signals given in parts (a) and (b). By superposition, its Fourier transform is

$$X_d(f) = X_a(f) - X_b(f)$$

$$= A\left[\frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}\right]$$

$$= \frac{-j4Af}{\alpha^2 + (2\pi f)^2}$$

<u>Problem 4-6</u>

(a) The Fourier transform integral for this signal becomes

$$X_a(f) = \int_{-\infty}^{\infty} te^{-\alpha t}u(t)e^{-j2\pi ft}dt = \int_0^{\infty} te^{-\alpha t}e^{-j2\pi ft}dt$$

$$= \int_0^{\infty} te^{-(\alpha + j2\pi f)t}dt = \left.\frac{te^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\right|_0^{\infty} + \frac{1}{(\alpha + j2\pi f)}\int_0^{\infty} e^{-(\alpha + j2\pi f)t}dt$$

$$= -\left.\frac{e^{-(\alpha + j2\pi f)t}}{(\alpha + j2\pi f)^2}\right|_0^{\infty} = \frac{1}{(\alpha + j2\pi f)^2}$$

(b) To evaluate this Fourier transform integral, we need the tabulated integral

$$\int t^2 e^{at}dt = \frac{e^{at}}{a^3}(a^2t^2 - 2at + 2)$$

For this signal, the Fourier transform integral is

$$X_b(t) = \int_{-\infty}^{\infty} t^2 u(t)u(1 - t)e^{-j2\pi ft}dt = \int_0^1 t^2 e^{-j2\pi ft}dt$$

$$= \left\{\frac{e^{-j2\pi ft}}{(-j2\pi f)^3}[(-j2\pi f)^2t^2 - 2(-j2\pi f)t + 2]\right\}_0^1$$

$$= \left\{\frac{1}{j2\pi f}\left[\frac{1}{j\pi f} - 1\right]e^{-j2\pi f} + \frac{2}{(j2\pi f)^3}\left[1 - e^{-j2\pi f}\right]\right\}$$

(c) The integral for this Fourier transform becomes

$$X_c(f) = \int_0^1 e^{-(\alpha + j2\pi f)t}dt = -\left.\frac{e^{-(\alpha + j2\pi f)t}}{\alpha + j2\pi f}\right|_0^1 = \frac{1 - e^{-(\alpha + j2\pi f)}}{\alpha + j2\pi f}$$

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Problem 4-8

(a) Using the time delay and superposition theorems, we obtain

$$X_1(f) = \frac{1}{2}[e^{j2\pi f} + e^{j\pi f} + e^{-j\pi f} + e^{-j2\pi f}]$$

$$= \cos(2\pi f) + \cos(\pi f)$$

(b) Use the transform pairs

$$2W\,sinc(2Wt) \leftrightarrow \Pi\left(\frac{f}{2W}\right)$$

and

$$u(t) \leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

with convolution in the frequency domain to obtain

$$X_2(f) = \Pi(f) * \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= \frac{1}{2}\Pi(f) + \int_{-\infty}^{\infty} \Pi(\lambda)\frac{1}{j2\pi(\lambda - f)}d\lambda$$

$$= \frac{1}{2}\Pi(f) + \frac{1}{j2\pi}\int_{-1/2}^{1/2} \frac{d\lambda}{\lambda - f}$$

$$= \frac{1}{2}\Pi(f) + \frac{1}{j2\pi}\ln\left|\frac{2f-1}{2f+1}\right|$$

(c) Note that $\mathrm{sgn}(t) = 2\,u(t) - 1$. Therefore

$$x_3(t) = 2\,sinc(t)u(t) - sinc(t)$$

$$= 2x_2(t) - sinc(t)$$

Using the results of part (b), we obtain

$$X_3(f) = \Pi(f) + \frac{1}{j\pi}\ln\left|\frac{2f-1}{2f+1}\right| - \Pi(f)$$

$$= \frac{1}{j\pi}\ln\left|\frac{2f-1}{2f+1}\right|$$

(d) The unit step makes the part of this signal to the left of $t = 0$ zero, so we have

$$X_4(f) = \mathcal{F}[\exp(-t)u(t)] = \frac{1}{1 + j2\pi f}$$

(e) Write this signal as

$$x_5(t) = e^{-|t|}[2u(t) - 1] = 2x_4(t) - e^{-|t|}$$

Thus, its Fourier transform is

^ 1 1 $j4\pi f$

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Problem 4-9

For $x(t)$ real

$$X(f) = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft) - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\,dt$$

$$= X_R(f) + jX_I(f)$$

where the first integral is the real part and the second integral is the imaginary part. The magnitude is

$$|X(f)| = \sqrt{X_R^2(f) + X_I^2(f)} = |X(-f)|$$

where the eveness of the magnitude function is obvious because the real and imaginary parts are squared. The phase function is

$$\theta(f) = \underline{/X(f)} = \tan^{-1}\left[\frac{X_I(f)}{X_R(f)}\right] = -\tan^{-1}\left[\frac{X_I(-f)}{X_R(-f)}\right] = -\theta(-f)$$

The last equation holds because the imaginary part of $X(f)$ is an odd function of f (the dependence is through a sine function) and the real part is an even function. Thus, the ratio of imaginary and real parts is odd. Furthermore, the arctangent is an odd function, making the overall result odd.

Problem 4-12

(a) See Problem 4-11for $G_a(f)$. The energy contained in frequencies less than α/π Hz is

$$E[|f| < \alpha/\pi] = 2\int_0^{\alpha/\pi} \frac{A^2\,df}{\alpha^2 + (2\pi f)^2}$$

$$= \frac{A^2}{\alpha\pi}\int_0^2 \frac{dx}{1 + x^2}$$

$$= \frac{A^2}{\alpha\pi}\tan^{-1}(2)$$

The total energy is $A^2/2\pi$, so the fraction of total energy is

$$Fraction\ of\ total = \frac{2}{\pi}\tan^{-1}(2) = \frac{2(1.107)}{\pi} = 70.5\%$$

(b) For this case,

$$E[|f| < \alpha/2\pi] = 2\int_0^{\alpha/2\pi} \frac{A^2\,df\ \cdot}{\alpha^2 + (2\pi f)^2}$$

$$= \frac{A^2}{\alpha\pi}\int_0^1 \frac{dx}{1 + x^2}$$

$$= \frac{A^2}{\alpha\pi}\tan^{-1}(1) = \frac{A^2}{4\alpha}$$

The fraction of total power is

$$Fraction\ of\ total = \frac{2}{\pi}\tan^{-1} = \frac{1}{2} = 50\%$$

Problem 4-17

The amplitude spectrum of the recorded signal is

$$|X(f)| = A\Pi\left(\frac{f-5050}{9900}\right)$$

It is played back at half the speed as recorded. Thus, the playback time is twice the record time. Therefore, $a = 0.5$ and the playback spectrum is

$$|Y(f)| = (A/0.5)\Pi\left(\frac{f/0.5-5050}{9900}\right) = 2A\Pi\left(\frac{f-2525}{4950}\right)$$

This spectrum has a maximum frequency of 5000 Hz, or half that of the record spectrum.

Problem 4-20

(a) The signal can be expressed as

$$x_a(t) = \Lambda(t+1) + \Lambda(t-1)$$

The first derivative is

$$\frac{dx_a(t)}{dt} = \Pi(t+1.5) - \Pi(t+0.5) + \Pi(t-0.5) - \Pi(t-1.5)$$

The second derivative is

$$\frac{dx_a^2(t)}{dt^2} = \delta(t+2) - 2\delta(t+1) + 2\delta(t) - 2\delta(t-1) + \delta(t-2)$$

Now use superpostion and the transform pair given in the problem statement to obtain

$$\mathcal{F}\left\{\frac{dx_a^2(t)}{dt^2}\right\} = (j2\pi f)^2 X_a(f)$$

$$= e^{j4\pi f} - 2e^{j2\pi f} + 2 - 2e^{-j2\pi f} + e^{-j4\pi f}$$

$$= 2\cos(4\pi f) - 4\cos(2\pi f) + 2$$

$$= -8\sin^2(\pi f)\cos(2\pi f)$$

Therefore

$$X_a(f) = \frac{-8\sin^2(\pi f)\cos(2\pi f)}{-4\pi^2 f^2}$$

$$= 2sinc^2(f)\cos(2\pi f)$$

(b) This signal can be written as

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$$x_b(t) = -\Lambda(t+1) + \Lambda(t-1)$$

Two derivatives gives a sum of impulses:

$$\frac{dx_b^2(t)}{dt^2} = -\delta(t+2) + 2\delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

The Fourier transform of the second derivative is

$$\mathcal{F}\left\{\frac{dx_b^2(t)}{dt^2}\right\} = (j2\pi f)^2 X_b(f)$$

$$= -e^{j4\pi f} + 2e^{j2\pi f} - 2e^{-j2\pi f} + e^{-j4\pi f}$$

$$= -2j\sin(4\pi f) + 4j\sin(2\pi f)$$

$$= -2j\sin(4\pi f) + 4j\sin(2\pi f)$$

The Fourier transform of the original signal is

$$X_b(f) = -\frac{j}{\pi f}[2sinc(2f) - 2sinc(4f)]$$

Clearly the first transform is real and even, as it should be, because of the eveness of the signal. The second on is imaginary and odd, as it should be, because of the oddness of the signal.