Electric Circuits II

Two-Port Circuits
Interconnected Two-Port Circuits

Lecture #44

The material to be covered in this lecture is as follows:

- Terminated Two-Port circuit
- Terminal Behavior
- The Six characteristics of the terminated two-port circuit in terms of z parameters
- Interconnected Two-Port Circuits

After finishing this lecture you should be able to:

- Analyze The Terminated Two-Port circuit
- Determine The characteristics of the terminated circuit in terms of z parameters
- Recognize the different Interconnection of the Two-Port Circuits
- Analyze The Cascade Connection

Terminated Two-Port Circuit

- The Circuit is driven at port 1 and loaded at port 2
- A typically terminated two-port model is shown in Fig. 44-1

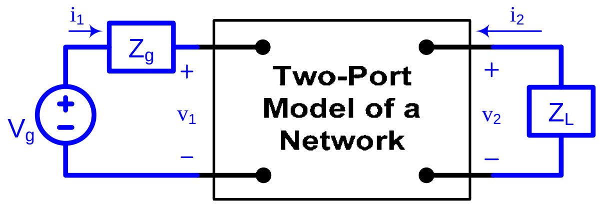

Fig. 44-1 A Terminated Two-Port Model

- Z_g represents the internal impedance of the source
- Z_L represents the Load impedance
- V_g represents the internal voltage of the source
- Analysis of this circuit involves expressing the terminal currents and voltages as function of V_g, Z_L, and Z_g.

Terminal Behavior:

Six characteristics of the terminated two-port circuit define its terminal behavior.

1. The input impedance $Z_{in} = \frac{V_1}{I_1}$ or admittance $Y_{in} = \frac{I_1}{V_1}$
2. The output current I_2
3. Thevenin voltage and impedance (V_{Th}, Z_{Th}) with respect to port 2

4. The current gain $\frac{I_2}{I_1}$
5. The voltage gain $\frac{V_2}{V_1}$
6. The current gain $\frac{V_2}{V_g}$

The six characteristics in Terms of the z Parameters:

- We develop the expressions using the z-parameters to model the two-port portion of the circuit.
- Expressions involving other parameters y, a, b, h and g can be found in tables in text books.
- The derivation of any one of the desired expressions involves the algebraic manipulation of the two-port equations along with the two constraint equations.
- These four equations using z parameters are:

i. $V_1 = z_{11}I_1 + z_{12}I_2$ (44-1)

ii. $V_2 = z_{21}I_1 + z_{22}I_2$ (44-2)

iii. $V_1 = V_g - I_1Z_g$ (44-3)

iv. $V_2 = -I_2Z_L$ (44-4)

To find the input impedance $Z_{in} = \frac{V_1}{I_1}$ we proceed as follows:

In (44-2) we replace V_2 from (44-4) we solve for I_2 we get: $I_2 = \dfrac{-z_{21}I_1}{Z_L + z_{22}}$ (44-5)

We then substitute in (44-1) and solve for Z_{in} we get: $Z_{in} = z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_L}$ (44-6)

To find I_2 we first solve (44-5) for I_1 after replacing V_1 with the RHS of (44-3) the result is:

$$I_1 = \frac{V_g - z_{12}I_2}{Z_g + z_{11}} \qquad (44\text{-}7)$$

We now substitute (44-7) into (44-5) and solve for I_2: $I_2 = \dfrac{-z_{21}V_g}{(Z_g + z_{11})(Z_L + z_{22}) - z_{12}z_{21}}$ (44-8)

The Thevenin voltage with respect to port 2 equals V_2 when $I_2 = 0$. Therefore:

$$V_2\big|_{I_2=0} = z_{21}I_1 = z_{21}\frac{V_1}{z_{11}} \qquad (44\text{-}9)$$

But $V_1 = V_g - I_1Z_g$, and $I_1 = \dfrac{V_g}{Z_g + z_{11}}$ when $I_2=0$; therefore by substitution into (44-9) the open circuit value of V_2 is:

$$V_2\big|_{I_2=0} = V_{Th} = \frac{z_{21}}{Z_g + z_{11}} V_g \qquad (44\text{-}10)$$

The Thevenin or output impedance is the ratio $Z_{Th} = \frac{V_2}{I_2}$ when V_g is zero (short circuit). Thus (44-3) becomes $V_1 = -I_1 Z_g$ (44-11)

Substituting gives $I_1 = \frac{-z_{12} I_2}{Z_g + z_{11}}$ (44-12)

Therefore $\left.\frac{V_2}{I_2}\right|_{V_g=0} = Z_{Th} = z_{22} - \frac{z_{12} z_{21}}{Z_g + z_{11}}$ (44-13)

The current gain comes directly from (44-5): $\frac{I_2}{I_1} = \frac{-z_{21}}{Z_L + z_{22}}$ (44-14)

To derive the voltage gain $\frac{V_2}{V_1}$ we replace I_2 in (44-2) with its value from (44-4):

$$V_2 = z_{21} I_1 + z_{22}\left(\frac{-V_2}{Z_L}\right) \qquad (44\text{-}15)$$

Next we solve (44-2) for I_1 in terms of V_1 and V_2:

$$z_{11}I_1 = V_1 - z_{12}\left(\frac{-V_2}{Z_L}\right)$$

Or

$$I_1 = \frac{V_1}{z_{11}} + \frac{z_{12}V_2}{z_{11}Z_L} \qquad (44\text{-}16)$$

Replace I_1 in (44-15) and solve:

$$\frac{V_2}{V_1} = \frac{z_{21}Z_L}{z_{11}Z_L + z_{11}z_{22} - z_{12}z_{21}} = \frac{z_{21}Z_L}{z_{11}Z_L + \Delta z} \qquad (44\text{-}17)$$

To derive the voltage gain $\frac{V_2}{V_g}$ we first find I_1 in terms of V_1 and V_2 by combining (44-1), (44-3) and (44-4):

$$I_1 = \frac{V_g}{z_{11} + Z_g} + \frac{z_{12}V_2}{Z_L\left(z_{11} + Z_g\right)} \qquad (44\text{-}18)$$

Then using (44-3) and (44-18) with (44-2) we derive an expression involving only V_2 and V_g:

$$V_2 = \frac{z_{21}z_{12}V_2}{Z_L\left(z_{11} + Z_g\right)} + \frac{z_{21}V_g}{z_{11} + Z_g} - \frac{z_{22}}{Z_L}V_2 \qquad (44\text{-}19)$$

After manipulation we get:

$$\frac{V_2}{V_g} = \frac{z_{21} Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12} z_{21}} \qquad (44\text{-}20)$$

Example 44-1

The two-port circuit of Fig. 44-2 is described in terms of its b parameters, the values of which are:
$b_{11} = -20$, $b_{12} = -3k\Omega$, $b_{21} = -2mS$, and $b_{22} = -0.2$

a) Find the phasor voltage V_2
b) Find the average power delivered to the $5k\Omega$ load
c) Find the average power delivered to the input port
d) Find the load impedance for maximum average power transfer
e) Find the maximum average power delivered to the load in (d)

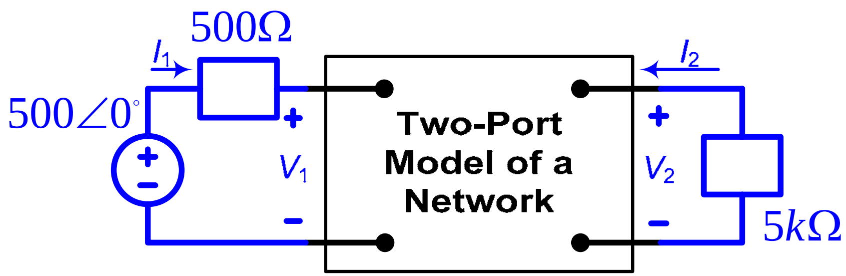

Fig. 44-2 The Circuit for Example 44-1

Solution:

a) To find V_2, we can either use (44-4) or from the voltage gain in (44-20), using the latter approach and conversion tables:

$$\Delta b = (-20)(-0.2) - (-3000)(-0.002) = -2$$

$$\frac{V_2}{V_g} = \frac{z_{21}Z_L}{(z_{11}+Z_g)(z_{22}+Z_L) - z_{12}z_{21}}$$

$$= \frac{\Delta b Z_L}{b_{12} + b_{11}Z_g + b_{22}Z_L + b_{21}Z_gZ_L} = \frac{(-2)(5000)}{-3000 + (-20)(500) + (-0.2)(5000) + (-0.002)(500)(5000)}$$

$= \frac{10}{19}$ Then, $V_2 = \frac{10}{19}500 = 263.16\angle 0^{\circ} V$

b) The average power delivered to the 5000Ω load is

$$P_2 = \frac{263.16^2}{2 \times 5000} = 6.93W$$

c) To find average power delivered to the input port, we first find the input impedance Z_{in},

From tables, $Z_{in} = \frac{b_{22}Z_L + b_{12}}{b_{11} + b_{21}Z_L} = \frac{(-0.2)(5000) - 3000}{(-0.002)(5000) - 20} = 133.33\Omega$

Therefore I_1 is

$$I_1 = \frac{500}{500 + 133.33} = 789.47mA$$

The average power delivered to the input port is $P_1 = \frac{0.78947^2}{2} 133.33 = 41.55W$

d) The total impedance for maximum power transfer Z_L is the conjugate of Thevenin impedance.

$$Z_{Th} = \frac{b_{11}Z_g + b_{12}}{b_{22} + b_{21}Z_g} = \frac{(-20)(500) - 3000}{(-0.002)(500) - 0.2} = 10833.33\Omega$$

Therefore $Z_L = Z_{Th}^* = 10833.33\Omega$

e) To find the maximum average power delivered to the load, we first find V_2 from the gain expression when $Z_L = 10833.33\Omega$

$\frac{V_2}{V_1} = 0.8333$ thus $V_2 = 0.8333 \times 500 = 416.67V$ and $P_{2\max} = \frac{1}{2}\frac{416.67^2}{10833.33} = 8.01W$

Interconnected Two-Port Circuits

Two-port circuits may be interconnected in five ways:

1) In Cascade Fig. 44-3a
2) In Series Fig. 44-3b
3) In Parallel Fig. 44-3c
4) In Series-Parallel Fig. 44-3d
5) In Parallel-Series Fig. 44-3e

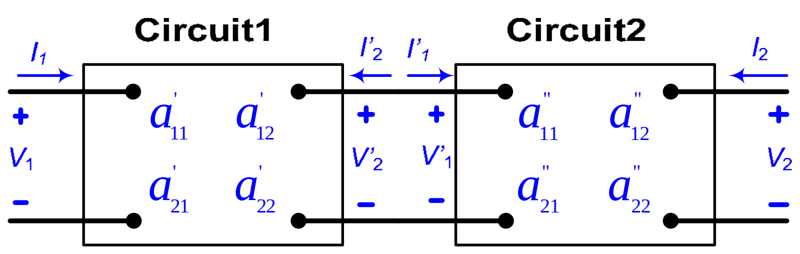

Fig. 44-4 Two-Port Cascade Connection

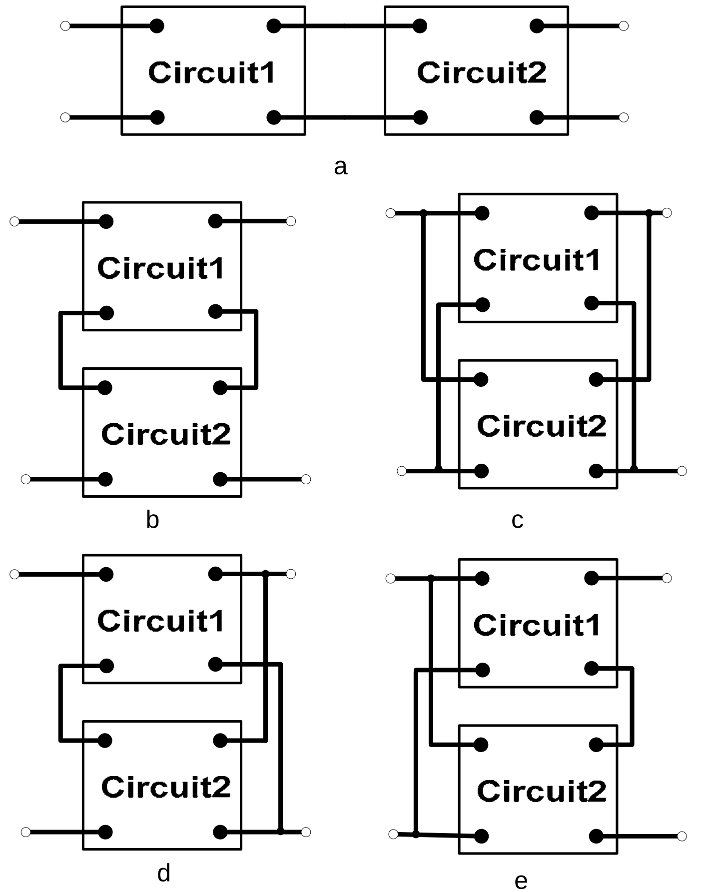

Fig. 44-3 Five Basic Interconnections (a) Cascade (b) Series (c) Parallel (d) Series-Parallel (e) Parallel-Series

- We analyze only the Cascade connection because it occurs frequently in the modeling of large systems.
- The a-parameters are best suited for describing cascade connection
- We seek the pair of equations:

$$V_1 = a_{11}V_2 - a_{12}I_2 \qquad (44\text{-}1)$$

$$I_1 = a_{21}V_2 - a_{22}I_2 \qquad (44\text{-}2)$$

From Fig. 44-4 we have

$$V_1 = a'_{11}V'_2 - a'_{12}I'_2 \qquad (44\text{-}3)$$

$$I_1 = a'_{21}V'_2 - a'_{22}I'_2 \qquad (44\text{-}4)$$

From interconnection $V'_2 = V'_1$ and $I'_2 = -I'_1$, then substituting yields:

$$V_1 = a'_{11}V'_1 - a'_{12}I'_1 \qquad (44\text{-}5)$$

$$I_1 = a'_{21}V'_1 - a'_{22}I'_1 \qquad (44\text{-}6)$$

From the second circuit

$$V'_1 = a''_{11}V_2 - a''_{12}I_2 \qquad (44\text{-}7)$$

$$I'_2 = a''_{21}V_2 - a''_{22}I_2 \qquad (44\text{-}8)$$

By substitution we generate

$$V_1 = \left(a'_{11}a''_{11} + a'_{12}a''_{12}\right)V_2 - \left(a'_{11}a''_{12} + a'_{12}a''_{22}\right)I_2 \qquad (44\text{-}9)$$

$$I_1 = \left(a'_{21}a''_{11} + a'_{22}a''_{21}\right)V_2 - \left(a'_{21}a''_{12} + a'_{22}a''_{22}\right)I_2 \qquad (44\text{-}10)$$

By comparison we get the desired expressions:

$$a_{11} = a'_{11}a''_{11} + a'_{12}a''_{12} \qquad (44\text{-}11)$$

$$a_{12} = a'_{11}a''_{12} + a'_{12}a''_{22} \qquad (44\text{-}12)$$

$$a_{21} = a'_{21}a''_{11} + a'_{22}a''_{21} \qquad (44\text{-}13)$$

$$a_{22} = a'_{21}a''_{12} + a'_{22}a''_{22} \qquad (44\text{-}14)$$

Self Test 44:

Find the transmission parameters for the circuit in Fig. 44-5

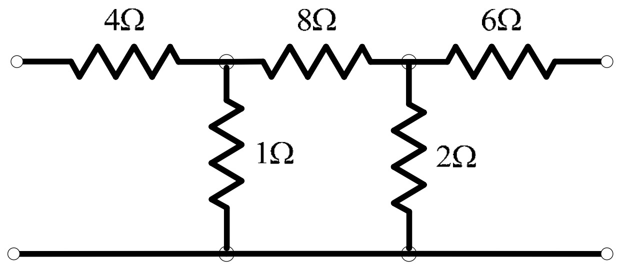

Fig. 44-5 The Circuit for Self Test 44

Answer:

$$[T] = [T_1][T_2] = \begin{bmatrix} 5 & 44 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0.5 & 4 \end{bmatrix} = \begin{bmatrix} 27 & 206\Omega \\ 5.5S & 42 \end{bmatrix}$$

i.e. $a_{11} = 27 \quad a_{12} = 206\Omega \quad a_{21} = 5.5S \quad a_{22} = 42$