Electric Circuits II

Two-Port Circuits
Hybrid and Transmission Parameters

Lecture #43

The material to be covered in this lecture is as follows:

- The Two-Port Hybrid parameters
- The Two-Port Transmission parameters
- Relationships Among the Two-Port Parameters
- Reciprocal Two-Port Circuits

After finishing this lecture you should be able to:

- Determine the Two-Port Hybrid Parameters
- Determine the Two-Port Transmission Parameters
- Derive all the Other Sets from a Known Set of Parameters
- Recognize Reciprocal and Symmetric Two-Port Circuits

Hybrid parameters

- The z and y parameters of a two-port network do not always exist.
- There is a need for developing another set of parameters.

Hybrid Parameters (h-parameters):

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2 \qquad (43\text{-}1)$$

or in matrix form:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = [h] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \qquad (43\text{-}2)$$

The h-parameters are very useful for describing electronic devices such as transistors. It is much easier to measure experimentally the h-parameters of such devices than to measure their z or y parameters. In fact an ideal transformer for example does not have z-parameters as it is impossible to express the voltages in terms of the currents or vice versa. See Fig. 43-1

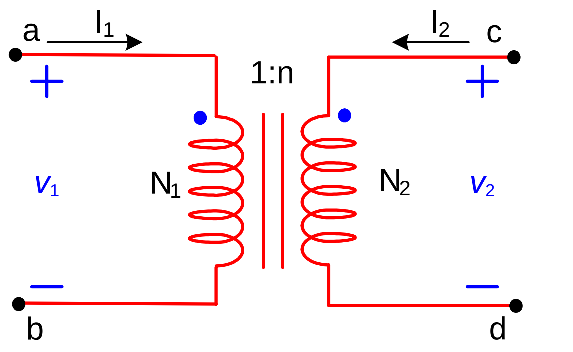

Fig. 43-1 An Ideal Transformer has no z-parameters

$$V_1 = \frac{1}{n} V_2$$
$$I_1 = -n I_2$$

The values of the parameters can be evaluated by setting $I_1=0$ (input port open-circuited) or $V_2=0$ (output port short-circuited). Thus,

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}, \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}, \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \qquad (43\text{-}3)$$

The parameters h_{11}, h_{12}, h_{21}, h_{22} represent an impedance, a voltage gain, a current gain, and an admittance respectively. This is why the h-parameters are called the hybrid parameters:

- h_{11}= short-circuit input impedance
- h_{12}= Open-circuit reverse voltage gain
- h_{21}= short-circuit forward current gain
- h_{22}= Open-circuit output admittance

The following example illustrates the determination of the h-parameters for a resistive circuit.

Example 43-1

Find the hybrid parameters for the two-port network of Fig. 43-2

Solution:

To find h_{11} and h_{21}, we short circuit the output port and connect a current source I_1 to the input port as shown in Fig. 43-3a

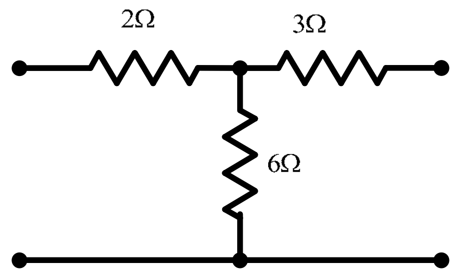

Fig. 43-2 Circuit for Example 43-1

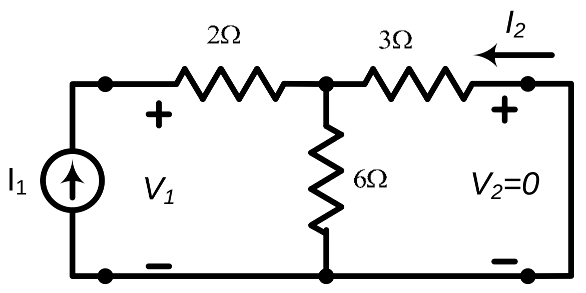

Fig. 43-3a Circuit for Example 43-1

$V_1 = I_1\left(2+3\|6\right) = 4I_1$ hence $h_{11} = \frac{V_1}{I_1} = 4\Omega$ and by current division, $-I_2 = \frac{6}{6+3}I_1 = \frac{2}{3}I_1$ hence $h_{21} = \frac{I_2}{I_1} = -\frac{2}{3}$

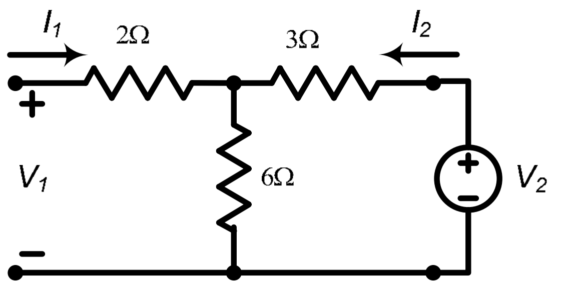

Fig. 43-3b Circuit for Example 43-1

To find h_{12} and h_{22}, we open circuit the input port and connect a voltage source I_2 to the input port as shown in Fig. 43-3b. By voltage division

$V_1 = \frac{6}{6+3}V_2 = \frac{2}{3}V_2$ hence $h_{12} = \frac{V_1}{V_2} = \frac{2}{3}$ and,

$V_2 = I_2\left(6+3\right) = 9I_2$ hence $h_{22} = \frac{I_2}{V_2} = \frac{1}{9}S$

Hybrid Parameters (g-parameters):

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2 \qquad (43\text{-}4)$$

or in matrix form:

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = [g] \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \qquad (43\text{-}5)$$

The values of the g-parameters can be evaluated by setting $V_1=0$ (input port short-circuited) or $I_2=0$ (output port open-circuited). Thus,

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0}, \; g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$
$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}, \; g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} \qquad (43\text{-}6)$$

The g-parameters or inverse hybrid parameters are:

- g_{11}= Open-circuit input admittance
- g_{12}= Short-circuit reverse current gain
- g_{21}= Open-circuit forward voltage gain
- g_{22}= Short-circuit output impedance

The following example illustrates the determination of the g-parameters.

Example 43-2

Find the g parameters as a function of s for the circuit of Fig. 43-4

Solution:

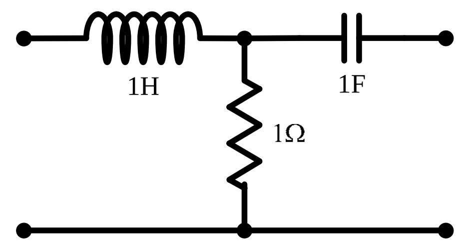

Fig. 43-4 Circuit for Example 43-2

In the s domain, we have $1H \Rightarrow sL = s$, $1F \Rightarrow \frac{1}{sC} = \frac{1}{s}$

To get g_{11} and g_{21}, we open circuit the output port and connect a voltage source V_1 to the input port as shown in Fig. 43-4a

$I_1 = \frac{V_1}{s+1}$ or $g_{11} = \frac{I_1}{V_1} = \frac{1}{s+1}$

By voltage division, $V_2 = \frac{1}{s+1} V_1$ or $g_{21} = \frac{V_2}{V_1} = \frac{1}{s+1}$

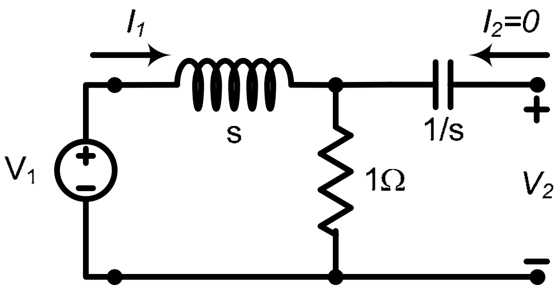

Fig. 43-4a Circuit for Example 43-2

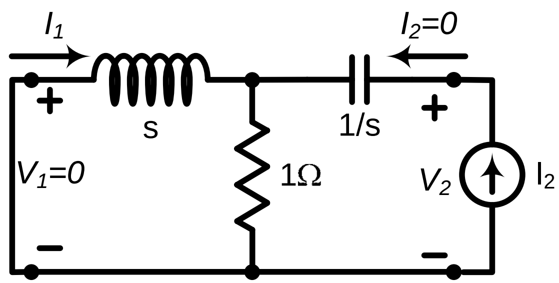

Fig. 43-4b Circuit for Example 43-2

To get g_{12} and g_{22}, we short circuit the input port and connect a current source I_2 to the output port as shown in Fig. 43-4b.

By current division, $I_1 = -\frac{1}{s+1} I_2$ or $g_{12} = \frac{I_1}{I_2} = -\frac{1}{s+1}$

Also, $V_2 = I_2\left(\frac{1}{s} + s \| 1\right)$ or $g_{22} = \frac{V_2}{I_2} = \frac{1}{s} + \frac{s}{s+1} = \frac{s^2+s+1}{s(s+1)}$

Thus, $[g] = \begin{bmatrix} \frac{1}{s+1} & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s^2+s+1}{s(s+1)} \end{bmatrix}$

Transmission parameters

As we have seen in Equs. (42-5) and (42-6) that we reproduce here for convenience:

Transmission Parameters (a-parameters): $$\begin{aligned} V_1 &= a_{11} V_2 - a_{12} I_2 \\ I_1 &= a_{21} V_2 - a_{22} I_2 \end{aligned} \qquad (43\text{-}7)$$

Inverse Transmission Parameters (b-parameters): $$\begin{aligned} V_2 &= b_{11} V_1 - b_{12} I_1 \\ I_2 &= b_{21} V_1 - b_{22} I_1 \end{aligned} \qquad (43\text{-}8)$$

where we express the input variables in terms of the output variables or the output variables in terms of the input variables.
In matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = [T] \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad (43\text{-}9)$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = [t] \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} \quad (43\text{-}10)$$

Where

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}, a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$
$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0}, a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0} \quad (43\text{-}11)$$

$$b_{11} = \left.\frac{V_2}{V_1}\right|_{I_1=0}, b_{12} = -\left.\frac{V_2}{I_1}\right|_{V_1=0}$$
$$b_{21} = \left.\frac{I_2}{V_1}\right|_{I_1=0}, b_{22} = -\left.\frac{I_2}{I_1}\right|_{V_1=0} \quad (43\text{-}12)$$

The a-parameters represent:

- a_{11}= Open-circuit reverse voltage ratio
- a_{12}= Negative short-circuit transfer impedance
- a_{21}= Open circuit transfer admittance
- a_{22}= Negative short-circuit reverse current ratio

The b-parameters or inverse hybrid parameters are:

- b_{11}= Open-circuit voltage gain
- b_{12}= Negative short-circuit transfer impedance
- b_{21}= Open circuit transfer admittance
- b_{22}= Negative short-circuit current gain

- These parameters are useful in the analysis of transmission lines (such as cable and fiber) because they express sending-end variables in terms of receiving-end variables.
- They are also called ABCD parameters (for a) and abcd parameters (for b).
- They are used in the design of telephone systems, microwave networks, and radars.

Relationships among the Two-Port Parameters

- If we know one set of parameters we can derive all the other sets from the known set
- Table 43-1 shows the Parameter Conversion Table
- Let's derive the relationship between z and y and between z and a as an example.

To derive the relationship between z and y we first solve Eqs. (42-2) $\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned}$ for V_1 and V_2.

Then compare the coefficients of I_1 and I_2 in the resulting expressions to the coefficients of I_1 and I_2 in Eqs. (42-1) $\begin{aligned} V_1 &= z_{11}I_1 + z_{12}I_2 \\ V_2 &= z_{21}I_1 + z_{22}I_2 \end{aligned}$. From Eqs. (42-2) we have:

$$V_1 = \frac{\begin{vmatrix} I_1 & y_{12} \\ I_2 & y_{22} \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}} = \frac{y_{22}}{\Delta y} I_1 - \frac{y_{12}}{\Delta y} I_2 \quad (43\text{-}13) \qquad V_2 = \frac{\begin{vmatrix} y_{11} & I_1 \\ y_{21} & I_2 \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}} = \frac{y_{21}}{\Delta y} I_1 - \frac{y_{11}}{\Delta y} I_2 \quad (43\text{-}14)$$

comparing (43-13) and (43-14) with (42-1) shows:

$$z_{11} = \frac{y_{22}}{\Delta y} \quad (43\text{-}15) \qquad z_{12} = -\frac{y_{12}}{\Delta y} \quad (43\text{-}16) \qquad z_{21} = -\frac{y_{21}}{\Delta y} \quad (43\text{-}17) \qquad z_{22} = \frac{y_{11}}{\Delta y} \quad (43\text{-}18)$$

To find z-parameters as a function of the a-parameters, we rearrange Eqs. (43-7) in the form of Eqs. (42-1) and then compare the coefficients. From the second equation in (42-5):

$$V_2 = \frac{1}{a_{21}} I_1 + \frac{a_{22}}{a_{21}} I_2 \quad (43\text{-}19)$$

therefore, substituting Eq. (43-19) into the first equation of Eqs. (42-5) yields

$$V_1 = \frac{a_{11}}{a_{21}} I_1 + \left(\frac{a_{11} a_{22}}{a_{21}} - a_{12} \right) I_2 \quad (43\text{-}20)$$

From Eq. (43-20) $z_{11} = \dfrac{a_{11}}{a_{21}}$ (43-21) $\qquad z_{12} = \dfrac{\Delta a}{a_{21}}$ (43-22)

From Eq. (43-19) $z_{21} = \dfrac{1}{a_{21}}$ (43-23) $\qquad z_{22} = \dfrac{a_{22}}{a_{21}}$ (43-24)

Example 43-3

Two sets of measurements are made on a two-port resistive circuit. The first set is made with port 2 open, and the second set is made with port 2 short-circuited. The results are as follows:

Port 2 Open	Port 2 Short Circuited
$V_1 = 10mV$	$V_1 = 24mV$
$I_1 = 10\mu A$	$I_1 = 20\mu A$
$V_2 = -40V$	$I_2 = 1mA$

Find the h parameters of the circuit.

Solution:

From the short circuit test: $h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{24\times10^{-3}}{20\times10^{-6}} = 1.2k\Omega$ and $h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{10^{-3}}{20\times10^{-6}} = 50$

h_{12} and h_{22} cannot be obtained directly from the open circuit test. However we can obtain them from a-parameters either from a conversion table or through a similar procedure as seen in previous slides in this lecture. We get $h_{12} = \frac{\Delta a}{a_{22}}$, $h_{22} = \frac{a_{21}}{a_{22}}$ The a-parameters are

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{10\times10^{-3}}{-40} = -0.25\times10^{-3}$$

$$a_{21} = \frac{I_1}{V_2}\bigg|_{I_2=0} = \frac{10\times10^{-6}}{-40} = -0.25\times10^{-6}\,S$$

$$a_{12} = -\frac{V_1}{I_2}\bigg|_{V_2=0} = -\frac{24\times10^{-3}}{10^{-3}} = -24\Omega$$

$$a_{22} = -\frac{I_1}{I_2}\bigg|_{V_2=0} = -\frac{20\times10^{-6}}{10^{-3}} = -20\times10^{-3}$$ which give $\Delta a = a_{11}a_{22} - a_{12}a_{21} = 5\times10^{-6} - 6\times10^{-6} = -10^{-6}$

Therefore $h_{12} = \frac{\Delta a}{a_{22}} = \frac{-10^{-6}}{-20\times10^{-3}} = 5\times10^{-5}$ and $h_{22} = \frac{a_{21}}{a_{22}} = \frac{-0.25\times10^{-6}}{-20\times10^{-3}} = 12.5\mu S$

Reciprocal Two-Port Circuits

If a two-port circuit is reciprocal the following relationships exist among the port parameters:

$z_{12} = z_{21}$ (43-23) $y_{12} = y_{21}$ (43-24)

$a_{11}a_{22} - a_{12}a_{21} = \Delta a = 1$ (43-25) $b_{11}b_{22} - b_{12}b_{21} = \Delta b = 1$ (43-26)

$h_{12} = -h_{21}$ (43-27) $g_{12} = -g_{21}$ (43-28)

A two-port circuit is reciprocal if the interchange of an ideal voltage source at one port with an ideal ammeter at the other port produces the same ammeter reading.

Example 43-4

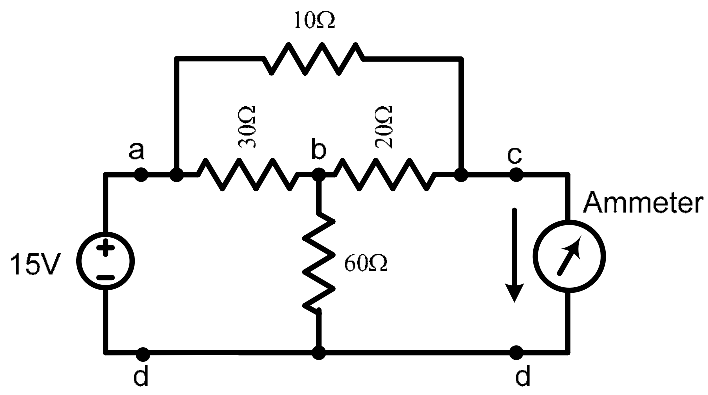

Fig. 42-4 A Reciprocal two-port Circuit

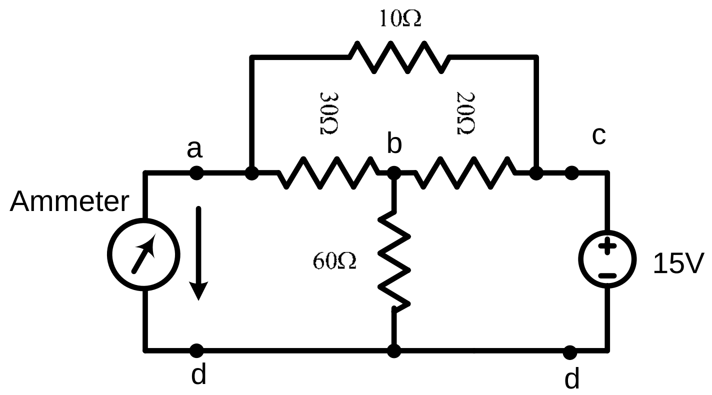

Fig. 42-5 The Circuit in Fig. 42-4 with the Voltage and Ammeter Interchanged

Consider the circuit of Fig. 43-5. When a voltage source of 15 V is applied to port ad, it produces a current of 1.75 A in the ammeter at port cd.
The ammeter current can be determined easily from V_{bd}

$$\frac{V_{bd}}{60}+\frac{V_{bd}-15}{30}+\frac{V_{bd}}{20}=0 \quad (43\text{-}29) \text{ and } V_{bd}=5V \text{ therefore } I=\frac{5}{20}+\frac{15}{10}=1.75A \quad (43\text{-}30)$$

If the voltage source and ammeter are interchanged the ammeter will still read 1.75 A. We verify this by solving the circuit of Fig. 43-6:

$$\frac{V_{bd}}{60}+\frac{V_{bd}}{30}+\frac{V_{bd}-15}{20}=0 \quad (43\text{-}31) \text{ therefore } V_{bd}=7.5V \text{ the current } I_{ad} \text{ is}$$

$$I_{ad} = \frac{7.5}{30} + \frac{15}{10} = 1.75A \qquad (43\text{-}32)$$

A reciprocal circuit is also symmetric, if its port can be interchanged without disturbing the values of the terminal currents and voltages. The following additional relationships will hold:

$z_{11} = z_{22}$ (43-33) $y_{11} = y_{22}$ (43-34)

$h_{11}h_{22} - h_{12}h_{21} = \Delta h = 1$ (43-35) $g_{11}g_{22} - g_{12}g_{21} = \Delta g = 1$ (43-36)

For a symmetric reciprocal circuit only two calculations or measurements are necessary to determine all the two-port parameters.

Self Test 43:

Two sets of measurements are made on a two-port network that is symmetric and reciprocal. The first set is made with port 2 open, and the second set is made with port 2 short-circuited as follows:

Port 2 Open	Port 2 Short Circuited
$V_1 = 95V$	$V_1 = 11.52V$
$I_1 = 5A$	$I_2 = -2.72A$

Calculate the z parameters of the two-port network.

Answer:

$z_{11} = z_{22} = 19\Omega$ $z_{12} = z_{21} = 17\Omega$