Electric Circuits II

Frequency Selective Circuits (Filters)

Bode Plots: Complex Poles and Zeros

Lecture #41

The material to be covered in this lecture is as follows:

- Bode Diagrams for Complex Poles and Zeros
- Straight-Line Amplitude Plots
- Correcting Straight-Line Amplitude Plots
- Phase Angle Plots

After finishing this lecture you should be able to:

- Use Bode Diagrams for Complex Poles and Zeros
- Draw Straight-Line Approximation of the Amplitude Plot for a Pair of Complex Poles (Zeros)
- Draw Straight-Line Approximation of the Phase Angle Plot
- Correct the plots for a Pair of Complex Poles (Zeros)

Bode Diagrams for Complex Poles and Zeros

- The complex poles and zeros of H(s) always appear in conjugate pairs
- Always combine the conjugate pair into a single quadratic term
- Once the rules for handling poles is understood, their application to zeros becomes apparent.

Let's consider $H(s) = \dfrac{K}{(s+\alpha - j\beta)(s+\alpha + j\beta)}$ (41-1)

The product $(s+\alpha - j\beta)(s+\alpha + j\beta)$ can be written as:

$(s+\alpha)^2 + \beta^2 = s^2 + 2\alpha s + \alpha^2 + \beta^2$ (41-2)

or $s^2 + 2\alpha s + \alpha^2 + \beta^2 = s^2 + 2\zeta\omega_n s + \omega_n^2$ (41-3)

where $\omega_n^2 = \alpha^2 + \beta^2$ (41-4) and $\zeta\omega_n = \alpha$ (41-5)

The term ω_n is the corner frequency of the quadratic factor

The term ζ is the damping coefficient of the quadratic factor,

- if $\zeta < 1$ the roots are complex
- if $\zeta > 1$ the roots are real,
- if $\zeta = 1$ this is the critical value.

For real roots we treat them as we have seen in previous lecture (Lec#40).

Assume $\zeta < 1$ then $H(s) = \dfrac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ (41-5)

$H(s)=\frac{K}{\omega_n^2}\frac{1}{1+2\zeta\left(\frac{s}{\omega_n}\right)+\left(\frac{s}{\omega_n}\right)^2}$ (41-6) in standard form will be $H(s)=\frac{K}{\omega_n^2}\frac{1}{1+\left(\frac{s}{\omega_n}\right)^2+2\zeta\left(\frac{s}{\omega_n}\right)}$ (41-7)

$H(j\omega)=\frac{K_0}{1-\left(\frac{\omega^2}{\omega_n^2}\right)+j\left(\frac{2\zeta\omega}{\omega_n}\right)}$ (41-8) where $K_0=\frac{K}{\omega_n^2}$

Let $u=\frac{\omega}{\omega_n}$ then $H(j\omega)=\frac{K_0}{1-u^2+j2\zeta u}$ (41-9) in polar form $H(j\omega)=\frac{K_0}{|1-u^2+j2\zeta u|\angle\beta_1}$ (41-10)

From which: $A_{dB}=20\log_{10}|H(jw)| =20\log_{10}K_0-20\log_{10}\left(\left(1-u^2\right)^2+\left(2\zeta u\right)^2\right)^{\frac{1}{2}}$ (41-11)

$-20\log_{10}\left(\left(1-u^2\right)^2+\left(2\zeta u\right)^2\right)^{\frac{1}{2}}=-10\log_{10}\left[u^4+2u^2\left(2\zeta^2-1\right)+1\right]$ (41-12)

and $\theta(\omega)=-\beta_1=-\tan^{-1}\frac{2\zeta u}{1-u^2}$ (41-13)

Approximate Amplitude Plots

$$A_{dB} = 20\log_{10} K_0 - 10\log_{10}\left[u^4 + 2u^2\left(2\zeta^2 - 1\right) + 1\right]$$

Because $u = \frac{\omega}{\omega_n}$ therefore $u \to 0\, as \quad \omega \to 0$ and $u \to \infty\, as \quad \omega \to \infty$ thus:

$$-10\log_{10}\left[u^4 + 2u^2\left(2\zeta^2 - 1\right) + 1\right] \to 0 \quad \text{as} \quad u \to 0$$

$$-10\log_{10}\left[u^4 + 2u^2\left(2\zeta^2 - 1\right) + 1\right] \to -40\log_{10} u \quad \text{as} \quad u \to \infty$$

We conclude that the approximate amplitude plot consists of two straight lines.

- For frequencies $\omega < \omega_n$ the straight line has a slope of 0dB
- For frequencies $\omega > \omega_n$ the straight line has a slope of -40dB/decade.

These two straight lines join on the 0dB axis at u=1 or $\omega = \omega_n$. Fig. 41-1 shows the straight line approximation for a quadratic factor with $\zeta < 1$.

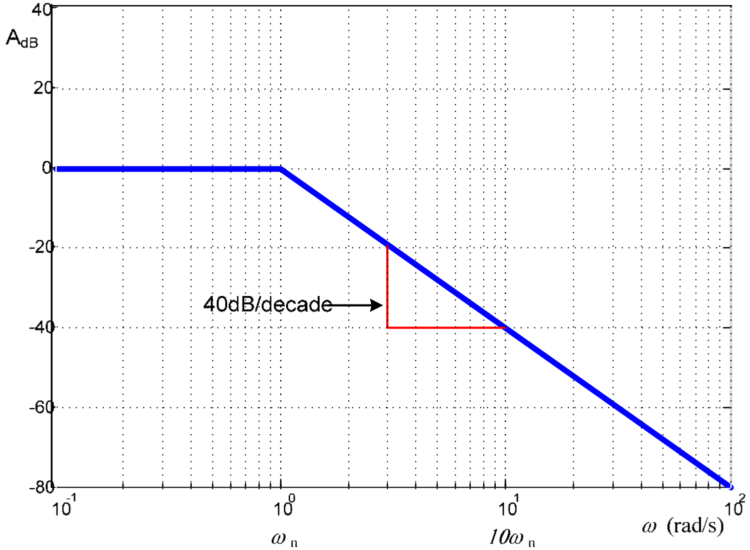

Fig. 41-1 the amplitude plot for a pair of complex poles

Correcting Straight-Line Amplitude Plots:

- Correcting the straight-line amplitude plot for a pair of complex poles is not as easy as correcting a first-order real pole, because the corrections depend on the damping coefficient ζ
- The straight-line amplitude plot can be corrected by locating four points on the actual curve:

  - Point 1 at ½ the corner frequency

    $\frac{\omega}{\omega_C}=\frac{1}{2} \quad =\frac{\omega}{\omega_n}=u$ then $\left[u^4+2u^2\left(2\zeta^2-1\right)+1\right]=\frac{1}{8}+\frac{1}{2}\left(2\zeta^2-1\right)+1=\zeta^2+\frac{5}{8}$

    $A_{dB}\left(\omega_n/2\right)=-10\log_{10}\left(\zeta^2+0.5625\right)$

  - Point 2 at peak amplitude frequency

    The amplitude peaks at $\omega_p=\omega_n\sqrt{1-2\zeta^2}=u$, thus $A_{dB}\left(\omega_p\right)=-10\log_{10}\left(4\zeta^2\left(1-\zeta^2\right)\right)$

  - Point 3 at the corner frequency $\frac{\omega}{\omega_n}=u=1$

    $A_{dB}\left(\omega_n\right)=-10\log_{10}4\zeta^2$ or $A_{dB}\left(\omega_n\right)=-20\log_{10}2\zeta$

  - Point 4 at zero amplitude frequency

    $\omega_0=\omega_n\sqrt{2\left(1-2\zeta^2\right)}=\sqrt{2}\omega p$

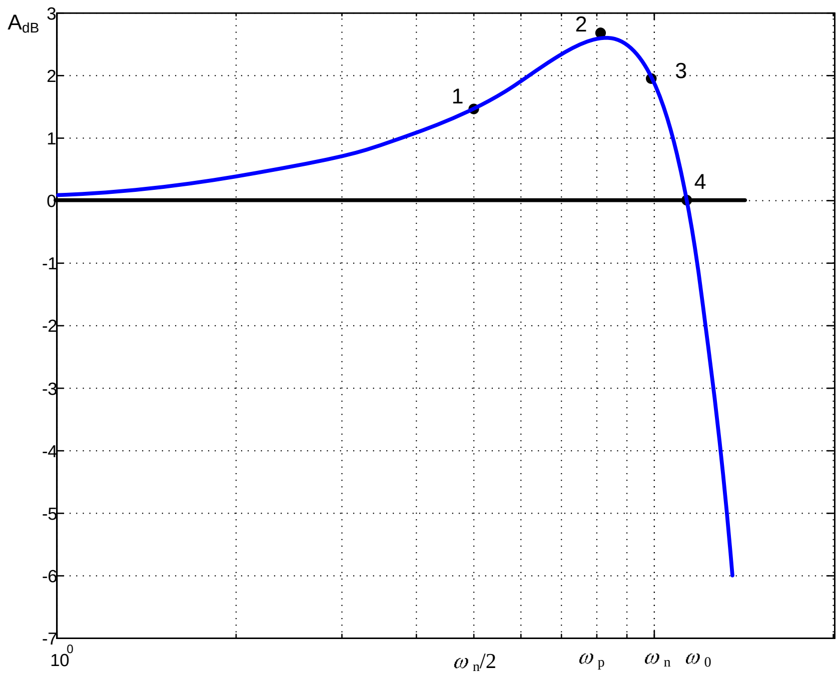

Fig. 41-2 Four points on the corrected amplitude plot for a pair of complex poles

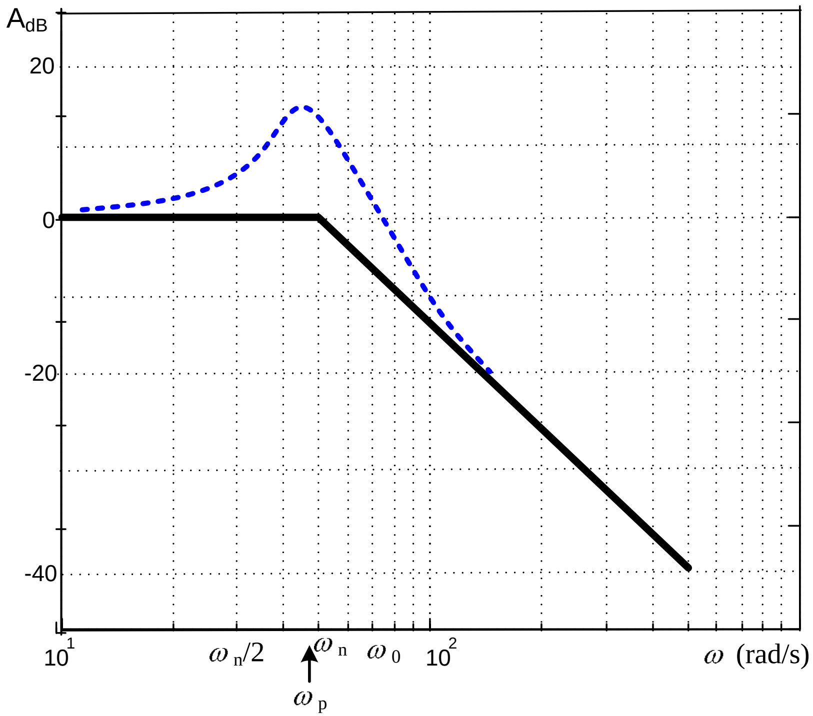

Fig. 41-4 the amplitude plot for Example 41-1

Example 41-1

Compute the transfer function for the circuit shown in Fig. 41-3

a) Find the corner frequency ω_n,
b) The value of K_0,
c) The damping coefficient ζ
d) Make a straight line amplitude plot ranging from 10 to 500 rad/s.
e) Sketch the actual corrected amplitude in dB at $\omega_n/2$, ω_p, ω_n, and ω_0.
f) Describe the type of filter represented by the circuit in Fig. 41-3

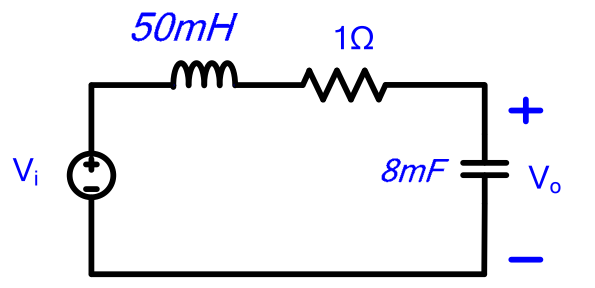

Fig. 41-3 The circuit for Example 41-1

Solution:

Transforming to s domain $H(s)=\dfrac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}=\dfrac{2500}{s^2+20s+2500}$

a) from the expression of H(s), $\omega_n^2=2500$, therefore $\omega_n=50\,rad/s$
b) By definition $K_0=\dfrac{2500}{\omega_n^2}$ or 1

c) The coefficient of s is $2\zeta\omega_n$ therefore $\zeta = \frac{20}{2\omega_n} = 0.20$

d) See Fig. 41-4

e) The actual amplitudes are

$$A_{dB}(\omega_n/2) = -10\log_{10}(0.6025) = 2.2dB$$

$$\omega_p = 50\sqrt{0.92} = 47.96 rad/s \qquad A_{dB}(\omega_p) = -10\log_{10}(0.16)(0.96) = 8.14dB$$

$$A_{dB}(\omega_n) = -20\log_{10}(0.4) = 7.96dB$$

$$\omega_0 = \sqrt{2}\omega_p = 67.82dB \qquad A_{dB}(\omega_0) = 0dB$$

Fig. 41-4 shows the corrected plot.

f) From the plot we see that this filter is a LPF, its cutoff frequency appears to be 55rad/s almost the same as that predicted by the straight-line approximation.

Phase Angle Plots

- The phase angle is zero at zero frequency
- The phase angle is -90^o at corner frequency
- The phase angle is -180^o at large frequency as $\omega \to \infty$

- For small values of ζ , the phase angle changes rapidly in the vicinity of the corner frequency.
- The line tangent to the phase angle curve at -90^o has a slope of -2.3/ζ rad/decade or -132/ζ degree/decade and it intersects the 0^o and -180^o lines at $u_1 = 4.81^{-\zeta}$ and $u_2 = 4.81^{\zeta}$ respectively.
- Fig. 41-5 depicts the straight-line approximation for $\zeta = 0.3$ and shows the actual phase angle plot.

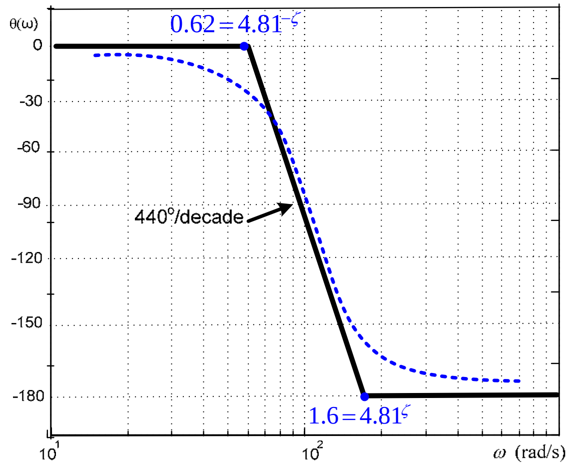

Fig. 41-5 Straight-Line Approximation of the Phase Angle for a pair of complex poles

Self Test 41:

The numerical expression for a transfer function is $H(s)=\frac{25\times10^8}{s^2+20\times10^3 s+25\times10^8}$

- ✓ Compute the corner frequency,
- ✓ the damping coefficient,
- ✓ frequencies when $H(j\omega)$ is unity,
- ✓ peak amplitude of $H(j\omega)$ in dB,
- ✓ frequency at which the peak occurs, and
- ✓ amplitude of $H(j\omega)$ at half the corner frequency.

Answer:

- ✓ $\omega_C = 50\,krad/s$,
- ✓ $\zeta = 0.2$,
- ✓ 0, $67.82\,krad/s$
- ✓ 8.14dB,
- ✓ $47.96\,krad/s$
- ✓ 2.20dB