Electric Circuits II

Frequency Selective Circuits (Filters)
Bandpass Filters

Lecture #38

The material to be covered in this lecture is as follows:

- Introduction to Bandpass filters
- Center Frequency, Bandwidth and Quality factor
- Bandpass Series RLC Circuit Analysis
- Relationship between different parameters

After finishing this lecture you should be able to:

- Understand the behavior of Bandpass filters
- Determine the different Bandpass filter parameters
- Relate each parameter in terms of other parameters
- Analyze Bandpass filter
- Design a simple Bandpass filter

Introduction to Bandpass filters

- Bandpass filters are those that pass voltages within a band of frequencies to the output while filtering out voltages at frequencies outside this band.
- Bandpass filters and Bandstop filters thus perform complementary functions in the frequency domain.
- These filters are characterized by same parameters
  - Two cutoff frequencies ω_{C1}, ω_{C2}
  - The center frequency ω_0
  - The bandwidth β
  - The quality factor Q
- Only two of these five parameters can be specified independently
- These parameters are defined in the same way for both types of filters. So we examine only the Bandpass filter at the beginning

Bandpass Filter: Center Frequency, Bandwidth, and Quality Factor

- The center frequency ω_0 is defined as the frequency for which a circuit's TF is purely real. It is also referred to as the resonant frequency.
- The center frequency is the geometric center of the passband, that is, $\omega_0 = \sqrt{\omega_{C1}\omega_{C2}}$ (38-1)
- The magnitude of the TF is maximum at the center frequency $H_{\max} = \left| H\left(j\omega_0 \right) \right|$ (38-2)
- The bandwidth β is the width of the passband $\beta = \omega_{C2} - \omega_{C1}$ (38-3)
- The quality factor Q is the ratio of the center frequency to the bandwidth $Q = \frac{\omega_0}{\beta}$ (38-4)

Analysis of the Bandpass Series RLC Circuit

Consider the Bandpass Circuit of Fig. 38-1. Using the same arguments used for Low-pass and High-pass filters, so as before changing the frequency of the source results in changes to the impedance of the capacitor and the inductor.

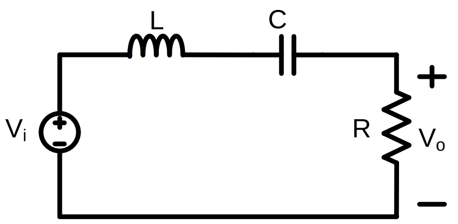

Fig. 38-1a A Series RLC Bandpass filter

- At $\omega = 0$ the impedance of the capacitor is $\frac{1}{j\omega C} = \infty$ so the capacitor behaves like an open circuit, and the impedance of the inductor is $j\omega L = 0$ so the inductor behaves like a short circuit. Fig. 38-1b depicts this result. Thus $V_o = 0$

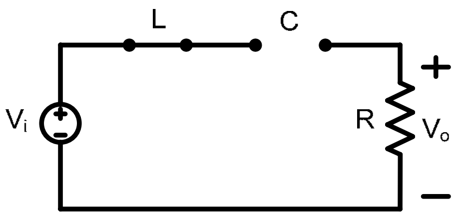

Fig. 38-1b Series RLC
Equivalent circuit for $\omega = 0$

Bandpass Series RLC Circuit Analysis cont.

- At $\omega = \infty$ the impedance of the capacitor is $\frac{1}{j\omega C} = 0$ so the capacitor behaves like a short circuit, and the impedance of the inductor is $j\omega L = \infty$ so the inductor behaves like an open circuit. Fig. 38-1c depicts this result. Thus $V_o = 0$ also.

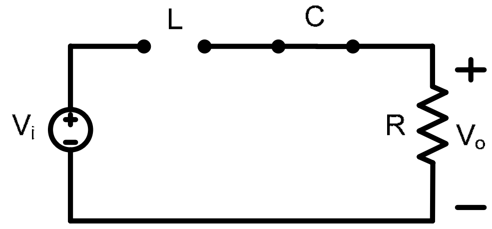

Fig. 38-1c Series RLC
Equivalent circuit for $\omega = \infty$

- Between these two extreme values, both the inductor and capacitor have finite impedances.
- At some special frequency in between, the center frequency ω_0, the two impedances cancel out causing the output voltage to equal the source voltage $V_o = V_i$. At ω_0 the series combination of L and C appears as a short circuit.
- The plot of the voltage magnitude ratio is shown in Fig. 38-2a. Note the ideal Bandpass filter plot is also overlaid on the plot of the RLC circuit.
- At very low frequencies the phase angle at the output maximizes at $+90^o$, and at very high frequencies the phase angle at the output reaches its negative maximum of -90^o as seen in Fig. 38-2b

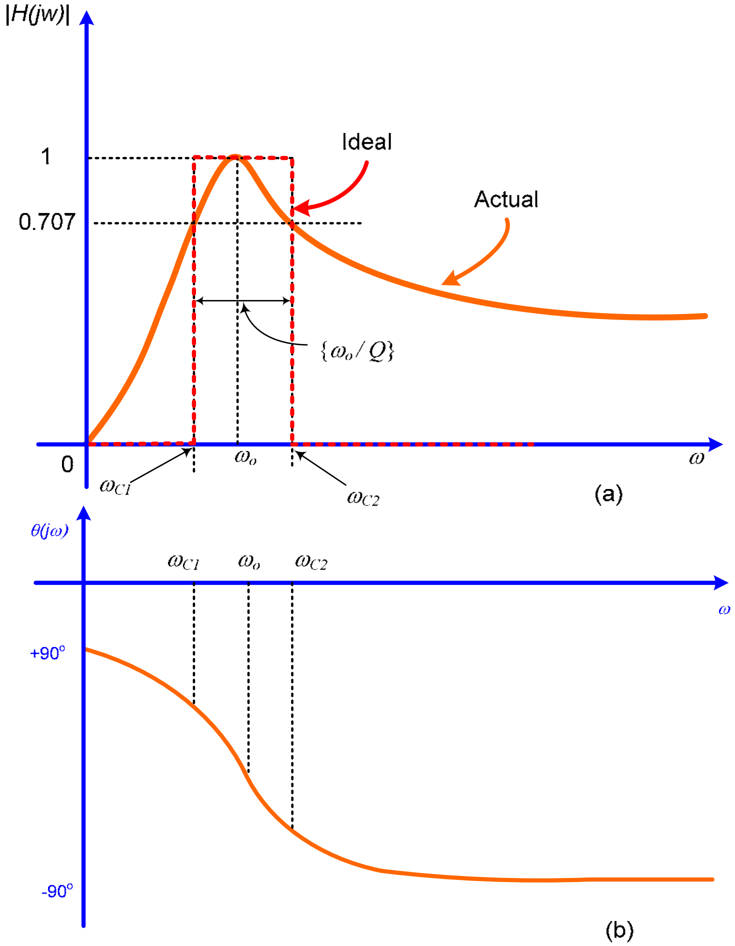

Fig. 38-2 The frequency response plot for the series RLC bandpass filter circuit in Fig. 38-1

Bandpass Series RLC Circuit Analysis Cont.

Let’s now examine the circuit quantitatively. Consider the s-domain equivalent circuit for the series RLC shown in Fig. 38-3 below

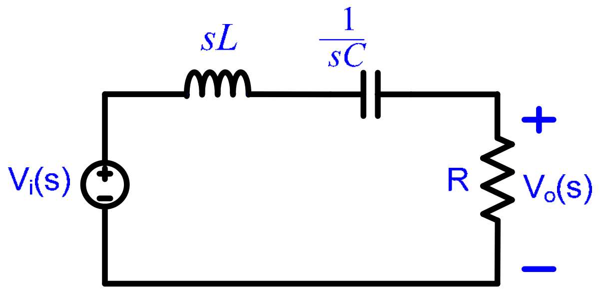

Fig. 38-3 s-domain equivalent for the circuit in Fig. 38-1a

Transfer function of the circuit:

$$H(s)=\frac{\left(\frac{R}{L}\right)s}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}} \qquad (38\text{-}5)$$

This equation gives magnitude and phase as before if we express it in polar form and substituting $s=j\omega$ we get:

$$\left|H(j\omega)\right| = \frac{\omega\frac{R}{L}}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2+\left(\omega\frac{R}{L}\right)^2}} \qquad (38\text{-}6)$$

$$\theta(j\omega) = 90^\circ - \tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC}-\omega^2}\right) \qquad (38\text{-}7)$$

We now calculate the five parameters that characterize this RLC bandpass filter:

Center Frequency: The TF in (38-5) will be purely real when $j\omega_0 L + \frac{1}{j\omega_0 C} = 0$ (38-8)

solving for ω_0 we get:

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad (38\text{-}9)$$

Cutoff Frequencies ω_{C1} and ω_{C2}: At the cutoff frequency $\left|H(j\omega_C)\right| = \frac{1}{\sqrt{2}} H_{\max}$ (38-10) where $H_{\max} = \left|H(j\omega_0)\right|$ we can calculate $H_{\max}$ by substituting equ. (38-9) into equ. (38-6)

$$H_{\max} = \frac{\omega_0 \frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \omega_0^2\right)^2 + \left(\omega_0 \frac{R}{L}\right)^2}}$$

$$= \frac{\sqrt{\frac{1}{LC}}\frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \frac{1}{LC}\right)^2 + \left(\sqrt{\frac{1}{LC}}\frac{R}{L}\right)^2}} = 1$$ Solving for ω_{C1} and ω_{C2}, using (38-9) we get:

$$\left|H(j\omega_C)\right| = \frac{1}{\sqrt{2}} = \frac{\omega_C \frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \omega_C^2\right)^2 + \left(\omega_C \frac{R}{L}\right)^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\left(\omega_C \frac{L}{R} - \frac{1}{\omega_C RC}\right)^2 + 1}} \qquad (38\text{-}11)$$

$$\pm 1 = \omega_C \frac{L}{R} - \frac{1}{\omega_C RC} \qquad (38\text{-}12)$$

rearranging and solving gives:

$$\omega_{C1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \qquad (38\text{-}13)$$

$$\omega_{C2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \qquad (38\text{-}14)$$

We can use these last two equations to confirm that ω_0 is the geometric mean of the two cutoff frequencies $\omega_0 = \sqrt{\omega_{C1}\omega_{C2}} = \sqrt{\frac{1}{LC}}$ (38-15)

We can compute also the bandwidth $\beta = \omega_{C2} - \omega_{C1} = \frac{R}{L}$ (38-16)

And the Quality factor $Q = \frac{\omega_0}{\beta} = \frac{\sqrt{1/LC}}{\left(R/L\right)} = \sqrt{L/CR^2}$ (38-17)

Note that we can express the cutoff frequencies in terms of center frequency and bandwidth as:

$$\omega_{C1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2} \qquad (38\text{-}18)$$

$$\omega_{C2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2} \qquad (38\text{-}19)$$

Example 38-1

a) Show that the RLC circuit in Fig. 38-4 is a bandpass filter by deriving an expression for the transfer function $H(s)$

b) Compute the center frequency ω_0

c) Calculate the Cutoff Frequencies ω_{C1} and ω_{C2}, the Bandwidth β and the Quality factor Q.

d) Compute values for R and L to yield a bandpass filter with $\beta = 200Hz$ a center frequency $\omega_0 = 5kHz$, using $5\mu F$ capacitor

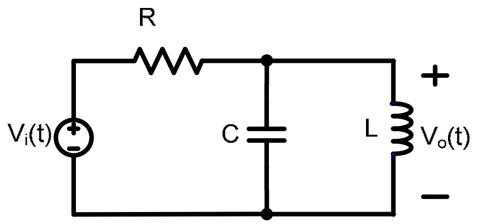

Fig. 38-4 Circuit for Example 38-1

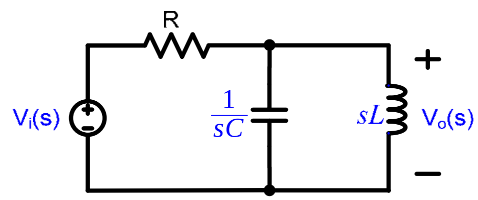

Fig. 38-5 s-domain equivalent for the Circuit of Fig. 38-4

Solution:

a) To derive the transfer function, consider the s-domain circuit in Fig. 38-5:

Using voltage division $Z_{eq}(s)=\dfrac{\frac{L}{C}}{sL+\frac{1}{sC}}$ Now, $H(s)=\dfrac{\frac{s}{RC}}{s^2+\frac{s}{RC}+\frac{1}{LC}}$

b) Center frequency: Substituting $s=j\omega$ we get

$$|H(j\omega)|=\frac{\frac{\omega}{RC}}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2+\left(\frac{\omega}{RC}\right)^2}}=\frac{1}{\sqrt{1+\left(\omega RC-\frac{1}{\omega\frac{L}{R}}\right)^2}}$$

the magnitude of this TF is maximum when the term $\left(\frac{1}{LC}-\omega^2\right)^2$ is zero.

Thus, $\omega_0 = \sqrt{\frac{1}{LC}}$ and $H_{\max} = \left| H\left(j\omega_0 \right) \right| = 1$

c) At the cutoff frequencies $\left| H\left(j\omega_C \right) \right| = \frac{1}{\sqrt{2}} H_{\max} = \frac{1}{\sqrt{2}}$ Substituting this constant in the magnitude equation above and simplifying, we get: $\left(\omega_C RC - \frac{1}{\omega_C \frac{L}{R}} \right) = \pm 1$ Solving gives:

$$\omega_{C1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC} \right)^2 + \left(\frac{1}{LC} \right)} \qquad \omega_{C2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC} \right)^2 + \left(\frac{1}{LC} \right)}$$

Bandwidth $\beta = \omega_{C2} - \omega_{C1} = \frac{1}{RC}$ Finally the Quality factor $Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2 C}{L}}$

Note that we can express the cutoff frequencies in terms of center frequency and bandwidth as seen in equations (38-18) and (38-19)

$$\omega_{C1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2} \right)^2 + \omega_0^2} \qquad \omega_{C2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2} \right)^2 + \omega_0^2}$$

Self Test 38:

Design a bandpass filter of the form in Fig. 38-1a, with $\omega_{C1} = 20.1kHz$, $\omega_{C2} = 20.3kHz$. Take $R = 20k\Omega$. Calculate L, C, and Q

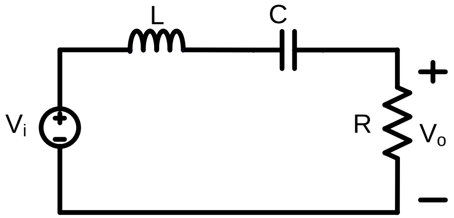

Fig. 38-1a A Series RLC Bandpass filter

Answer:

$L = 15.92H$, $C = 3.9\,pF$, $Q = 101$