Electric Circuits II

Frequency Selective Circuits (Filters)
Low and High Pass Filters

Lecture #37

The material to be covered in this lecture is as follows:

- Series RC Circuit as a Low Pass filter
- Relating The Frequency Domain to the Time Domain
- Series RC Circuit as High-Pass filter
- Series RC Circuit- Qualitative Analysis
- The Series RC Circuit- Quantitative Analysis

After finishing this lecture you should be able to:

- Understand the Behavior of High-Pass Filters
- Distinguish between Low-Pass and High-Pass Filters
- Relate The Cutoff Frequency to the Time Constant
- Analyze Qualitatively And Quantitatively A High-Pass Filter
- Design A Simple High-Pass Filter

The Series RC Circuit Low-Pass Filter

- The series RC Circuit shown in Fig. 37-1 also behaves as a low pass filter.

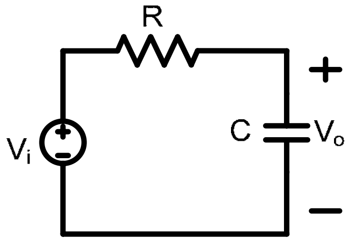


Fig. 37-1 A Series
RC Low-Pass filter

- Note that the circuit's output is defined as the output across the capacitor.
- As we did in the previous qualitative analysis in Lec36, we use three frequency regions to develop the behavior of the series RC circuit in Fig. 37-1:
  1. Zero Frequency $\omega = 0$ $\rightarrow$ impedance of the capacitor $Z_c = \infty$, the capacitor acts as an open circuit. The input and output voltages are the same as shown in Fig. 37-2a

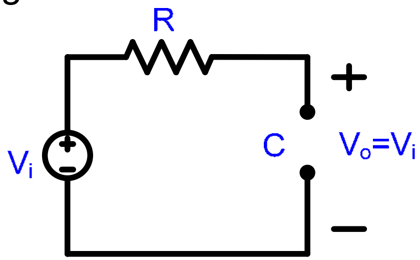


Fig. 37-2a A Series RC
Circuit at $\omega = 0$

2. Frequencies increasing from zero. The impedance of the capacitor $Z_c \ll R$, The output voltage is smaller than the source voltage which divides between R and C.

3. Infinite frequency $\omega = \infty \rightarrow$ impedance of the capacitor $Z_c = 0$, the capacitor acts as a short circuit. The output voltage is zero as shown in Fig. 37-2b

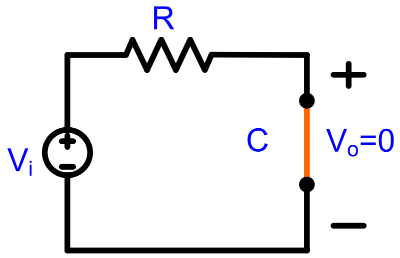


Fig. 37-2b A Series RC Circuit at $\omega = \infty$

Based on this analysis we see that the series RC circuit functions as a low-pass filter. The following example explores the quantitative analysis of the circuit.

Example 37-1

For the series RC circuit in Fig. 37-1 above:

- Find the transfer function $H(j\omega) = \frac{V_o}{V_i}$
- Determine the cutoff frequency in the circuit
- Choose values of R and C that will yield a low pass filter with $\omega_c = 3kHz$. Take $C = 1\mu F$.

Solution:

To derive the transfer function, consider the s-domain circuit in Fig. 37-3:

$$H(s)=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$ substituting $s=j\omega$ we get

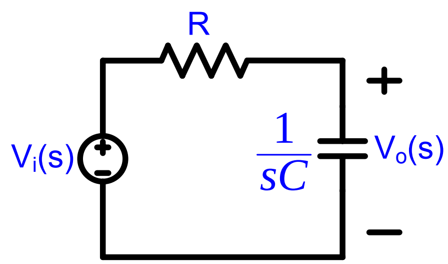


Fig. 37-3 s-domain equivalent for Fig. 37-1

$$\left|H(j\omega)\right|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$ The cutoff frequency can be found by

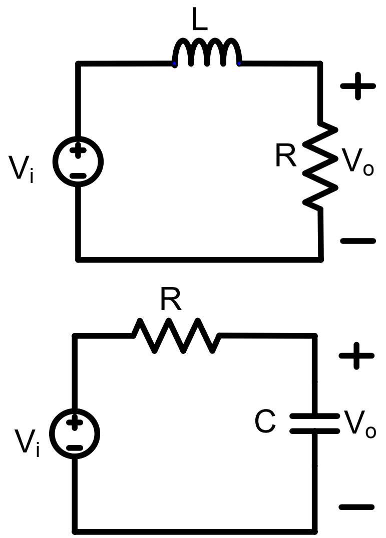
solving $\left|H(\omega_C)\right|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}=\frac{1}{\sqrt{2}}$ for ω_C we get: $\omega_C=\frac{1}{RC}$

Given $C=1\mu F$ then $R=\frac{1}{\omega_C C}=\frac{1}{(2\pi)\left(3\times10^3\right)\left(1\times10^{-6}\right)}=53\Omega$

Summary:

Fig. 37-4 summarizes the two low-pass filter circuits we have seen in Lec36 and this lecture. Look how similar in form their transfer functions are:

$$H(s)=\frac{\omega_C}{s+\omega_C} \qquad (37\text{-}1)$$



$$H(s)=\frac{\frac{R}{L}}{s+\frac{R}{L}}$$

$$\omega_C=\frac{R}{L}$$

$$H(s)=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

$$\omega_C=\frac{1}{RC}$$

Fig. 37-4 Two Low-Pass filters together with their Transfer Functions and Cutoff Frequencies

Relating the Frequency Domain to the Time Domain

An other important relationship is relating the time constant to the cutoff frequency.
Remember that for RL circuit the time constant is

$\tau=\frac{L}{R}$ and for an RC circuit the time constant is

$\tau=RC$

Compare the time constants to the cutoff frequencies for these circuits, notice that

$$\tau=\frac{1}{\omega_c} \qquad (37\text{-}2)$$

High Pass Filter
The Series RC Circuit – Qualitative Analysis

- Now we examine a High-Pass Filter, series RC Circuit shown in Fig. 37-5a
- The circuit's output is the voltage across R.
- Behavior of R will not change with changing frequency
- Behavior of capacitor C will change with changing frequency
- Recall that the impedance of a capacitor is $1/j\omega C$ at high frequencies the capacitor's impedance is very small compared with the resistor's impedance,
- The capacitor effectively functions as a short circuit. The term high frequencies thus refers to any frequencies for which $\omega C \gg R$
- The equivalent circuit for $\omega = \infty$ is shown in Fig. 37-5b
- At this frequency $V_i = V_o$ both in magnitude and phase angle
- Decreasing the frequency causes the capacitor's impedance to increase.
- Increasing the capacitor's impedance causes a corresponding increase in the magnitude of the voltage drop across C
- Increasing the capacitor's impedance causes a corresponding decrease in the magnitude of V_o the output voltage drop across R
- Increasing the capacitor's impedance also introduces a shift in phase angle between V_C the voltage across the capacitor and I_C the current in the capacitor

- This results in a phase angle difference between the input and output voltage, V_o Leads V_i as the frequency decreases this phase lead approaches 90°
- The capacitor effectively functions as an open circuit. The term Low frequencies thus refers to any frequencies for which $\omega C \ll R$
- The equivalent circuit for $\omega = 0$ is shown in Fig. 37-5c

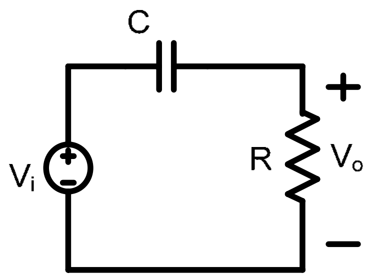


Fig. 37-5a A Series RC High-Pass filter

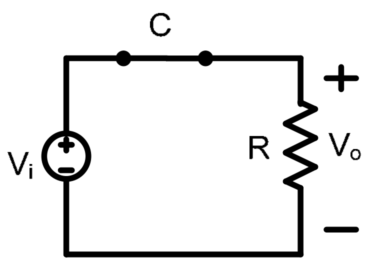


Fig. 37-5b RC Circuit Equivalent at $\omega = \infty$

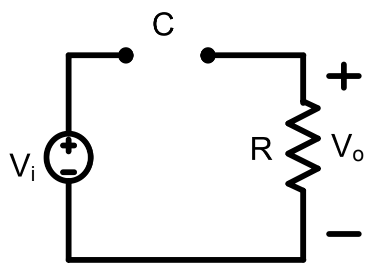


Fig. 37-5c RC Circuit Equivalent at $\omega = 0$

- Based on the behavior of the output voltage magnitude at low frequencies $V_o = 0$ at $\omega = 0$ this series RC circuit selectively passes high-frequency inputs to the output.
- This circuit response is shown in Fig. 37-6

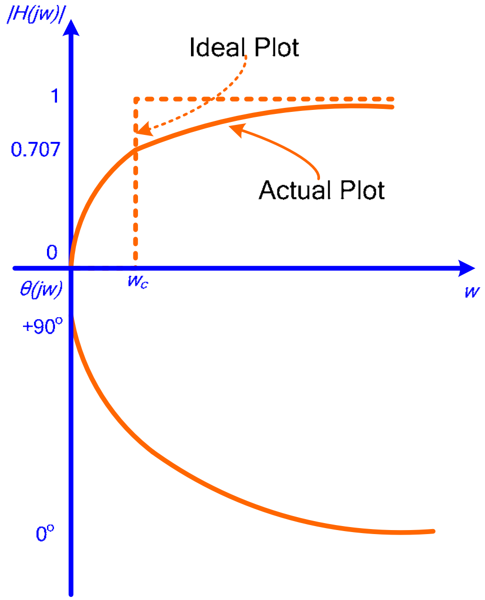


Fig. 37-6 Frequency Response for Series RC Circuit in Fig. 37-5a

The Series RC Circuit – Quantitative Analysis

- Consider the circuit of Fig. 37-7

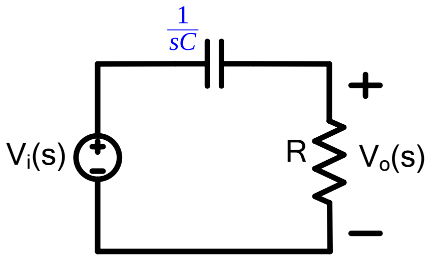


Fig. 37-7 s-domain equivalent for the Circuit of Fig. 37-5a

The voltage transfer function for this circuit is: $$H(s)=\frac{s}{s+\frac{1}{RC}} \qquad (37\text{-}3)$$

To study the frequency response we replace $s=j\omega$ in equ. (37-3): $$H(j\omega)=\frac{j\omega}{j\omega+\frac{1}{RC}} \qquad (37\text{-}4)$$

The Series RL Circuit – Quantitative Analysis (cont)

Equation (37-4) expressed in polar form gives the magnitude and phase:

$$\left|H(j\omega)\right| = \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}} \quad (37\text{-}5)$$

$$\theta(j\omega) = 90^\circ - \tan^{-1}\omega RC \quad (37\text{-}6)$$

Close examination of these equations provide quantitative support for the plots seen in Fig. 37-6.
We can now compute ω_C in terms of the circuit elements:

Recall from (36-1) that $\left|H(j\omega_C)\right| = \frac{1}{\sqrt{2}} H_{\max}$, and for high-pass filter, $H_{\max} = \left|H(j\infty)\right|$ as seen in Fig. 37-6 thus: $\left|H(j\omega_C)\right| = \frac{1}{\sqrt{2}} = \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$ (37-7),

Solving we get: $\omega_C = \frac{1}{RC}$ (37-8).

Not surprising it is the same as for Low-pass filter as the RC circuit has the same time constant whether configured as a low-pass or high-pass filter, as seen in equ. (37-2)

Example 37.2

Consider the circuit of Fig. 37-8, determine what type of filter is this and calculate its cutoff frequency. Take $R = 2k\Omega,\ L = 2H,$ and $C = 2\mu F$.

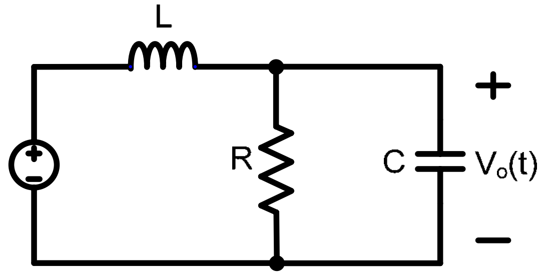


Fig. 37-8 Circuit for Example 37-2

Solution:

The transfer function

$$H(s) = \frac{V_o}{V_i} = \frac{R \| \frac{1}{sC}}{sL + R \| \frac{1}{sC}} \qquad (37\text{-}9)$$

But $R \| \frac{1}{sC} = \frac{R/sC}{R + 1/sC} = \frac{R}{1 + sRC}$ substituting gives:

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{R}{1 + sRC}}{sL + \frac{R}{1 + sRC}} = \frac{R}{s^2 RLC + sL + R} \qquad (37\text{-}10)$$

Example 37.2 cont.

or $H(\omega)=\dfrac{R}{-\omega^2 RLC+j\omega L+R}$ (37-11)

since $H(0)=1$ and $H(\infty)=0$, we conclude that it is a Low-Pass Filter.

Cutoff frequency:

$$\left|H(\omega_C)\right|=\frac{R}{\sqrt{\left(R-\omega_C^2 RLC\right)^2+\omega_C^2 L^2}}=\frac{1}{\sqrt{2}}$$

Or $$2=\left(1-\omega_C^2 LC\right)^2+\left(\frac{\omega_C L}{R}\right)^2$$

Substituting the values given, we obtain:

$2=\left(1-\omega_C^2 4\times10^{-6}\right)^2+\left(\omega_C\times10^{-3}\right)^2$ or $16\omega_C^4-7\omega_C^2-1=0$ solving we get : $\omega_C^2=0.5509$ or

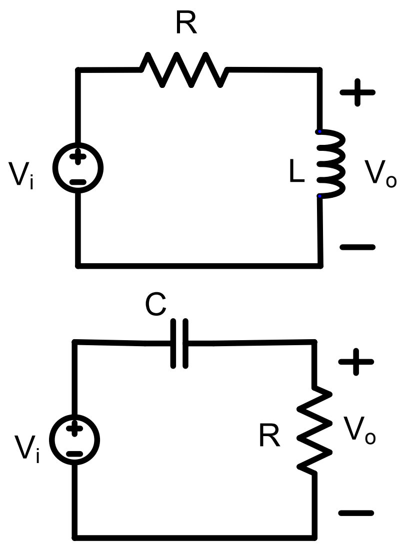
$$\omega_C=0.742$$

Assuming $\omega_C=0.742\,{}^{krad}/_{s}$, therefore the cutoff frequency is $\omega_C=742\,{}^{rad}/_{s}$

Summary:

Similarly to RC we can also form a High-Pass filter using RL Circuit and take the output across the inductor. Fig. 37-9 summarizes two High-pass filter circuits and their transfer functions. Look how similar in form their transfer functions are:

$$H(s)=\frac{s}{s+\omega_C} \qquad (37\text{-}12)$$



$$H(s)=\frac{s}{s+\frac{R}{L}}$$

$$\omega_C=\frac{R}{L}$$

$$H(s)=\frac{s}{s+\frac{1}{RC}}$$

$$\omega_C=\frac{1}{RC}$$

Fig. 37-9 Two High-Pass filters together with their Transfer Functions and Cutoff Frequencies

Self Test 37:

For the circuit in Fig. 37-10, obtain the transfer function $H(s)=\frac{V_o(\omega)}{V_i(\omega)}$ Identify the type of filter the circuit represents and determine the cutoff frequency. Take $R_1=100\Omega=R_2\ L=2mH,$

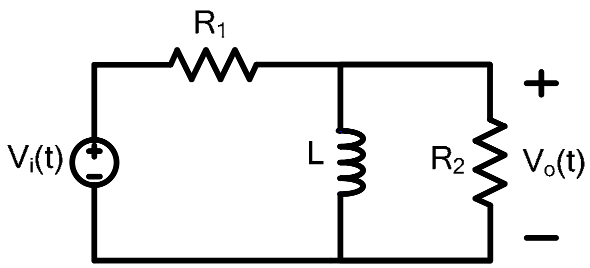


Fig. 37-10 Circuit for Self Test 37

Answer:

The transfer function $$H(s)=\frac{V_o}{V_i}=\frac{sR_2L}{R_1R_2+sL(R_1+R_2)} \qquad (37\text{-}13)$$

$$=\frac{\dfrac{R_2 sL}{R_2+sL}}{R_1+\dfrac{R_2 sL}{R_2+sL}}=\frac{\left(\dfrac{R_2}{R_1+R_2}\right)s}{s+\left(\dfrac{R_2}{R_1+R_2}\right)\dfrac{R_1}{L}}=\frac{Ks}{s+\omega_C} \qquad (37\text{-}14)$$

or $$H(\omega)=\frac{R_2}{(R_1+R_2)}\left(j\omega \middle/ j\omega+\frac{R_1R_2}{(R_1+R_2)L}\right) \qquad (37\text{-}15)$$

since $H(0)=0$ and $H(\infty)=1$, we conclude that it is a High-Pass Filter.

Cutoff frequency:

$$\omega_C=K\frac{R_1}{L}=\frac{R_1R_2}{(R_1+R_2)L}=25krad/s$$