Electric Circuits II

The Ideal Transformer

Lecture #34

The material to be covered in this lecture is as follows:

- The Ideal Transformer
- Determining the Voltage and Current Ratios
- Rules For Assigning Proper Algebraic Sign For Relating The Voltage And Current
- Impedance Matching

After finishing this lecture you should be able to:

- Understand the Behavior of Ideal Transformers
- Determine the polarity of the Voltage and Current Ratios
- Analyze Circuits Containing Ideal Transformers
- Use The Ideal Transformer For Impedance Matching

The Ideal Transformer

- An ideal transformer consists of two magnetically coupled coils having N_1 and N_2 turns respectively, and exhibiting these three properties:
  - The coefficient of coupling is unity $k = 1$,
  - The self-inductance of each coil is infinite $L_1 = L_2 = \infty$
  - The coil losses, due to parasitic resistance, are negligible.
- Understanding the behavior of ideal transformers begins with equation (33-11) which describes the impedance Z_{ab} we repeat this below

$$\mathbf{Z}_{ab} = \mathbf{Z}_{11} + \frac{\omega^2 M^2}{\mathbf{Z}_{22}} - \mathbf{Z}_S = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L} \qquad (34\text{-}1)$$

- Let us consider the same circuit as in Fig.33-1 of previous lecture

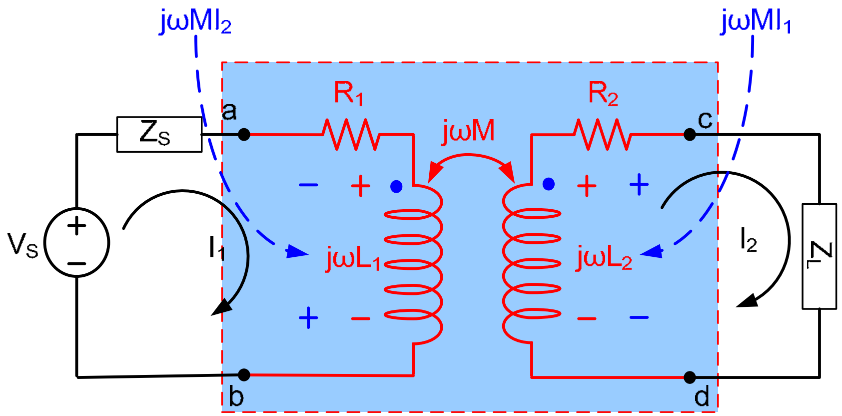


Fig.34-1 Frequency-domain model of a Linear Transformer

Exploring Limiting Values

To show how Z_{ab} changes when $k = 1, L_1$ and L_2 approach infinity we use the notation:

$$\mathbf{Z}_{22} = R_2 + R_L + j(\omega L_2 + \mathbf{X}_L) = R_{22} + j\mathbf{X}_{22} \qquad (34\text{-}2)$$

Then rearrange

$$\mathbf{Z}_{ab} = \overbrace{R_1 + \frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2}}^{R_{ab}} + j\overbrace{\left(\omega L_1 - \frac{\omega^2 M^2 X_{22}}{R_{22}^2 + X_{22}^2}\right)}^{X_{ab}} \qquad (34\text{-}3)$$

Before we let L_1 and L_2 increase, we write the coefficient as:

$$X_{ab} = \omega L_1 - \frac{\omega L_1 \omega L_2 X_{22}}{R_{22}^2 + X_{22}^2} = \omega L_1\left(1 - \frac{\omega L_2 X_{22}}{R_{22}^2 + X_{22}^2}\right) \qquad (34\text{-}4)$$

Exploring Limiting Values (Cont)

Using the fact that when, $k = 1$ then $M^2 = L_1 L_2$

Putting the term multiplying ωL_1, over a common denominator gives

$$X_{ab} = \omega L_1 \left(\frac{R_{22}^2 + \omega L_2 X_L + X_L^2}{R_{22}^2 + X_{22}^2} \right) \qquad (34\text{-}5)$$

Factoring ωL_2 out from numerator and denominator yields

$$X_{ab} = \frac{\omega^2 L_1 L_2}{\omega^2 L_2^2} \left(\frac{X_L + \left(R_{22}^2 + X_L^2\right) / \omega L_2}{R_{22}^2 / \omega^2 L_2^2 + \left[1 + \left(\frac{X_L}{\omega L_2} \right)^2 \right]} \right) \qquad (34\text{-}6)$$

Exploring Limiting Values (Cont)

$$X_{ab} = \frac{L_1}{L_2}\left(\frac{X_L + \left(R_{22}^2 + X_L^2\right)/\omega L_2}{\left(\frac{R_{22}}{\omega L_2}\right)^2 + \left[1 + \left(\frac{X_L}{\omega L_2}\right)^2\right]}\right) \quad (34\text{-}7)$$

As $k \rightarrow 1 \Rightarrow \frac{L_1}{L_2} \rightarrow$ constant value of $\left(\frac{N_1}{N_2}\right)^2$

The reason is that; as the coupling becomes extremely tight, the two permeances p_1 and p_2 become equal.

Therefore as $k \rightarrow 1, L_1 \rightarrow \infty, L_2 \rightarrow \infty$

$$X_{ab} = \left(\frac{N_1}{N_2}\right)^2 X_L \quad (34\text{-}8)$$

The same reasoning leads to simplification of the reflected resistance:

$$\frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2} = \frac{L_1}{L_2} R_{22} = \left(\frac{N_1}{N_2}\right)^2 R_{22} \quad (34\text{-}9)$$

Applying the results given by the two previous equations yields

$$Z_{ab} = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2 + \left(\frac{N_1}{N_2}\right)^2 (R_L + jX_L) \quad (34\text{-}10)$$

Exploring Limiting Values (Cont)

Comparing the two results we see that when $k \to 1, L_1 \to \infty, L_2 \to \infty$

- The transformer reflects the resistance of the secondary and load impedance to the primary side by a scaling factor equal to $\left(\frac{N_1}{N_2}\right)^2$ (the turns ratio squared).
- Hence we may describe the ideal transformer by two characteristics:

✓ $\left|\frac{v_1}{N_1}\right| = \left|\frac{v_2}{N_2}\right| \Rightarrow \left|\frac{v_1}{v_2}\right| = \frac{N_1}{N_2}$ (34-11)

✓ $\left|i_1 N_1\right| = \left|i_2 N_2\right| \Rightarrow \left|\frac{i_1}{i_2}\right| = \frac{N_2}{N_1}$ (34-12)

In the following section we show how to determine the voltage and current ratios.

Determining the Voltage and Current Ratios

- For unity coupling, $M^2 = L_1L_2$ or $M = \sqrt{L_1L_2}$ therefore (34-15) becomes: $V_2 = \sqrt{\frac{L_2}{L_1}}V_1$ (34-16)
- For unity coupling, the flux linking coil 1 is the same as the flux linking coil 2, so we need only one permeance to describe the self inductance of each coil, thus (34-16) becomes

  - $V_2 = \sqrt{\frac{N_2^2 p}{N_1^2 p}}V_1 = \frac{N_2}{N_1}V_1$ (34-17)

or

  - $\frac{V_1}{N_1} = \frac{V_2}{N_2}$ (34-18)

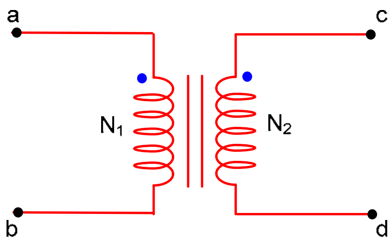


Fig.34-4 Ideal Transformer Symbol

Ampere-turn Ratio: Short Circuit Coil

Summing the voltages around the shorted coil of Fig.34-3 (b) yields

$$0 = -j\omega M\mathbf{I}_1 + j\omega L_2\mathbf{I}_2 \quad (34\text{-}19)$$

From which for k=1,

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{L_2}{M} = \frac{L_2}{\sqrt{L_1L_2}} = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1} \quad (34\text{-}20)$$

This is equivalent to $\mathbf{I}_1N_1 = \mathbf{I}_2N_2$ (34-21)

In practice, coils wound on a ferromagnetic material behave very much like an ideal transformer. Fig.34-4 shows the graphic symbol for an ideal transformer

Determining the Polarity of the Voltage and Current Ratios

- In the this section we show how to establish reference polarities for currents and voltages and remove the magnitude signs from equations (34-11) and (34-12)
- Rules for assigning proper algebraic sign for relating the voltage and current
  - o If the coil voltages v_1 and v_2 are both positive or negative at the dot-marked terminal, use a plus sign, otherwise, use a minus sign
  - o If the coil currents i_1 and i_2 are both directed into or out of the dot-marked terminal, use a minus sign, otherwise, use a plus sign
- The following four circuits illustrate these rules:

Fig.34-6 shows three ways to represent the same turn ratio of an ideal transformer (8 in this case)

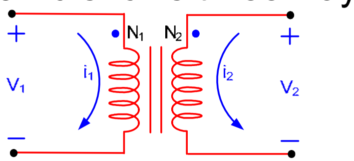


Fig.34-5 (a) Both Currents Entering and Both Voltages Positive at Dotted Terminals

$\frac{V_1}{N_1} = \frac{V_2}{N_2}$ $\quad I_1N_1 = -I_2N_2$

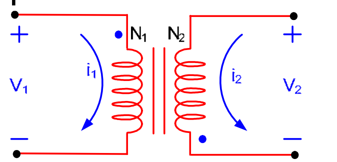
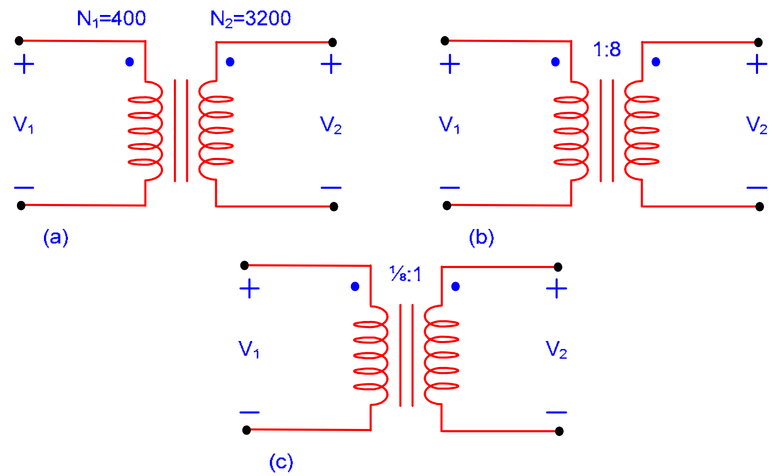


Fig.34-5 (b) i_1 Entering, i_2 leaving and v_1 Positive, v_2 Negative at Dotted Terminals

$\frac{V_1}{N_1} = -\frac{V_2}{N_2}$ $\quad I_1N_1 = I_2N_2$



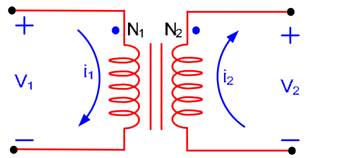


Fig.34-5 (c) i_1 Entering, i_2 leaving andBoth Voltages Positive at Dotted Terminals

$\frac{V_1}{N_1} = \frac{V_2}{N_2}$ $\quad I_1N_1 = I_2N_2$

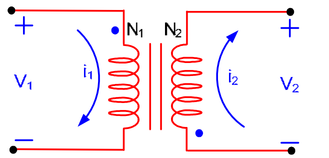


Fig.34-5 (d) Both Currents Entering and v_1 Positive, v_2 Negative at Dotted Terminals

$\frac{V_1}{N_1} = -\frac{V_2}{N_2}$ $\quad I_1N_1 = -I_2N_2$

Fig.34-6 Three Ways to Show the Same Turn Ratio

Example 34-1

The load impedance connected in Fig.34-7 consists of $Z_L = 0.2375 + j0.05\,\Omega$ and $Z_S = 0.25 + j2\,\Omega$. If $v_g = 2500\cos 400t$ find the steady state expressions for a) i_1, b) v_1, c) i_2, d) v_2

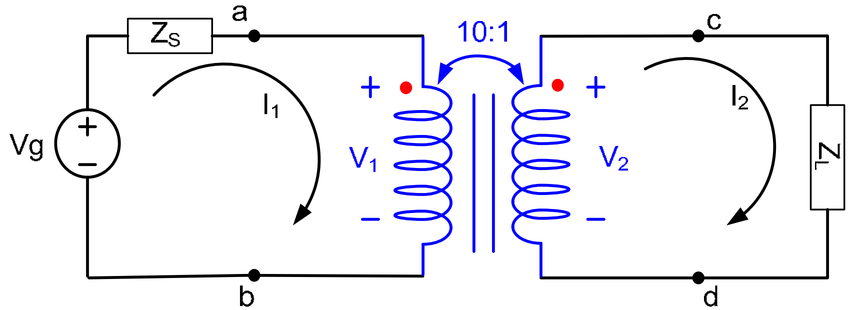


Fig.34-7 Circuit for Example 34-1

a)

$$\mathbf{V}_g = 2500\angle 0^\circ = (0.25 + j2)\mathbf{I}_1 + \mathbf{V}_1$$

and

$$\mathbf{V}_1 = 10\mathbf{V}_2 = 10\left[(0.2375 + j0.05)\mathbf{I}_2\right]$$

Because

$$\mathbf{I}_2 = 10\mathbf{I}_1$$

we have

$$\mathbf{V}_1 = 10\mathbf{V}_2 = 100(0.2375 + j0.05)\mathbf{I}_1$$

$$= (23.75 + j5)\mathbf{I}_1$$

Therefore

$$2500\angle 0^\circ = (24 + j7) I_1$$

or

$$\mathbf{I_1} = 100\angle\text{-}16.26^\circ \text{ A}$$

Thus the steady state expression for i1 is

$$i_1 = 100\cos(400t - 16.26^\circ) A$$

b)

$$\begin{aligned}\mathbf{V_1} &= 2500\angle 0^\circ - (100\angle\text{-}16.26^\circ)(0.25 + j2) \\ &= 2500 - 80 - j185 \\ &= 2420 - j185 = 2427.06\angle\text{-}4.37^\circ\end{aligned}$$

Hence

$$v_1 = 2427.06\cos(400t - 4.37^\circ) V$$

c)

$$\mathbf{I_2} = 10\mathbf{I_1} = 1000\angle\text{-}16.26^\circ \text{ A}$$

therefore

$$i_2 = 1000\cos(400t - 16.26^\circ) A$$

d)

$$\mathbf{V_2} = 0.1\mathbf{V_1} = 242.71\angle\text{-}4.37^\circ \text{ V}$$

therefore

$$v_2 = 242.71\cos(400t - 4.37^\circ) V$$

$$\mathbf{I_1} = 100\angle\text{-}16.26^\circ \text{ A}$$

Thus the steady state expression for i1 is $i_1 = 100\cos\left(400t - 16.26^\circ\right) A$

b)

$$\begin{aligned}\mathbf{V_1} &= 2500\angle 0^\circ - \left(100\angle\text{-}16.26^\circ\right)\left(0.25 + j2\right)\\ &= 2500 - 80 - j185\\ &= 2420 - j185 = 2427.06\angle\text{-}4.37^\circ\end{aligned}$$

Hence

$$v_1 = 2427.06\cos\left(400t - 4.37^\circ\right) V$$

c)

$$\mathbf{I_2} = 10\mathbf{I_1} = 1000\angle\text{-}16.26^\circ \text{ A}$$

therefore

$$i_2 = 1000\cos\left(400t - 16.26^\circ\right) A$$

d)

$$\mathbf{V_2} = 0.1\mathbf{V_1} = 242.71\angle\text{-}4.37^\circ \text{ V}$$

therefore

$$v_2 = 242.71\cos\left(400t - 4.37^\circ\right) V$$

The Use of Ideal Transformer for Impedance Matching

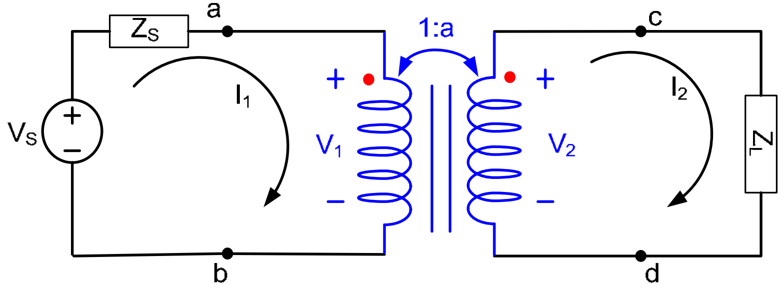


Fig.34-8 Coupling a Load to a Source by an Ideal Transformer

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{a} \qquad \text{and} \qquad \mathbf{I}_1 = a\mathbf{I}_2$$

Therefore

$$\mathbf{Z}_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{V}_2/a}{a\mathbf{I}_2} = \frac{1}{a^2}\frac{\mathbf{V}_2}{\mathbf{I}_2}$$

and $\mathbf{Z}_L = \frac{\mathbf{V}_2}{\mathbf{I}_2} \quad \Rightarrow \quad \mathbf{Z}_{in} = \frac{\mathbf{Z}_L}{a^2}$

Thus the ideal transformer's secondary coil reflects the load impedance back to the primary coil, with the scaling factor $\frac{1}{a^2}$

Z_{in} is greater or less than Z_L depends on the turns ratio a .

Self Test:

1) The source voltage in the phasor domain circuit in the Fig.34-9 below is $25\angle 0^\circ kV$. Find the amplitude and phase angle of V_2 and I_2.

$$V_2 = 1868.15\angle 142.39^\circ V$$
$$I_2 = 125\angle 216.87^\circ A$$

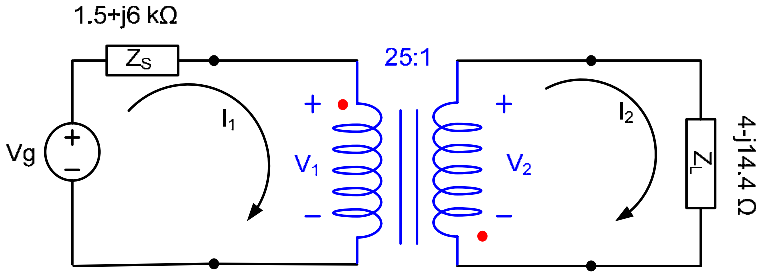


Fig.34-9 Circuit for Self Test

2) The output impedance of the amplifier is 192 Ω and the internal impedance of the speaker is 12 Ω. Determine the required turn ratio n to achieve impedance matching for maximum power transfer.

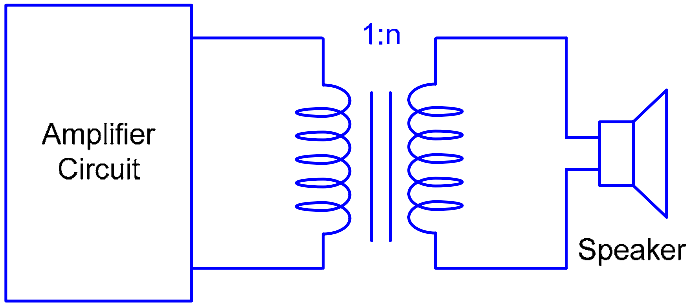


Fig.35-10 Matching Speaker to Amplifier

Answer: n=0.25