2. Mathematical Models of Systems (cont.)

Signal Flow Graph [SFG] Model

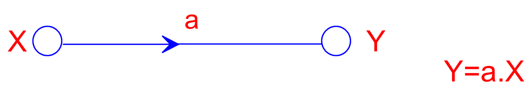
- For a complex system, , the block diagram reduction technique is cumbersome. An alternative method for determining the relationship between system variables has been developed by Mason and is based on a signal flow graph.
- A signal flow graph is a diagram that consists of nodes that are connected by branches. A node is assigned to each variable of interest in the system, and branches are used to relate the different variables.
- The main advantage for using SFG is that a straight forward procedure is available for finding the transfer function in which it is not necessary to move pickoff point around or to redraw the system several times as with block diagram manipulations.

SFG Algebra

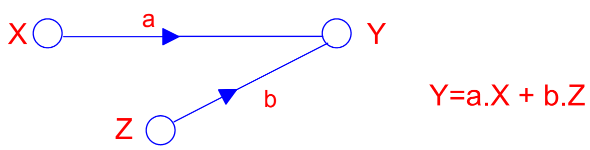
SFG: A diagram consisting of nodes and branches.

Node: A junction denoting a variable or a signal.

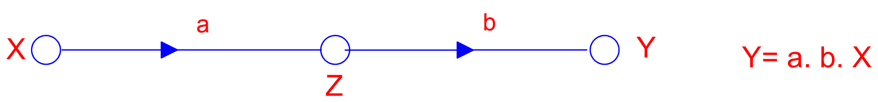
Branch:
- A unidirectional path that relates the dependency of an input and an output.
- Relation between variables are written next to the directional arrow.



- $\sum$of all signals entering a node is equal to the node

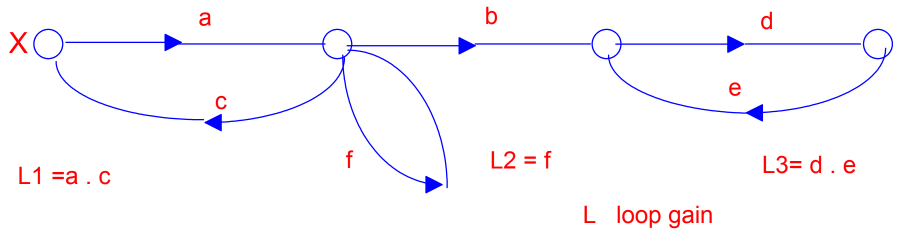


Path: A branch or a continuous sequence of branches that can be traversed from one node to another



Path gain = a.b

Loop: A closed path that originates and terminates on same node. Along the path no node is met twice



Non-Touching Loops: Loops with no common nodes

Examples: L1 and L2 are touching loops
L1 and L3 are non-touching loops, also L2 and L3 are non-touching loops.

Source: node having only outgoing branches

Sink: node having only incoming branches

MASON'S RULE

$$T = \sum_{k=1}^{N} \frac{P_k \Delta_k}{\Delta}$$

N	Number of paths from input to output
P	Gain of the k_{th} path from input to output
Δ	Determinant of the graph
Δ_k	Cofactor of the k_{th} path

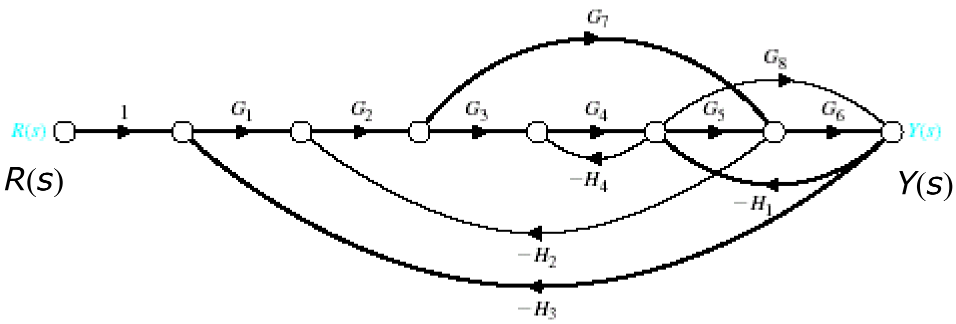
Summation is taken over all possible paths from input to output

$\Delta = 1$ - ($\sum$ all different loop gains)
+ ($\sum$ gain products of all combinations of 2 non-touching loops)
- ($\sum$ gain products of all combinations of 3 non-touching loops)
+ ($\sum$ gain products of all combinations of 4 non-touching loops)
- .....

$\Delta_k =$ value of Δ for that part of the SFG not touching the k_{th} forward path

Example

Find $\frac{Y(s)}{R(s)}$



Solution

Forward Paths =3

$P_1 = G_1G_2G_3G_4G_5G_6$; $P_2 = G_1G_2G_7G_6$ $P_3 = G_1G_2G_3G_4G_8$

Feedback loops

$L_1 = -G_2G_3G_4G_5H_2$; $L_2 = -G_5G_6H_1$; $L_3 = -G_8H_1$; $L_4 = -G_7H_2G_2$;
$L_5 = -G_4H_4$; $L_6 = -G_1G_2G_3G_4G_5G_6H_3$; $L_7 = -G_1G_2G_7G_6H_3$
$L_8 = -G_1G_2G_3G_4G_8H_3$

Loop L_5 does not touch loop L_4 or Loop L_7
Loop L_3 does not touch Loop L_4
All other loops touch

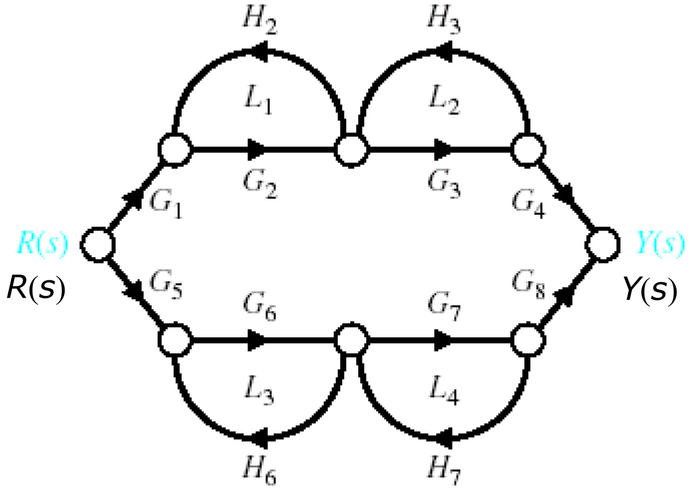
$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5L_7 + L_5L_4 + L_3L_4)$$

$$\Delta_1 = \Delta_3 = 1 ; \Delta_2 = 1 - L_5 = 1 + G_4H_4$$

$$\frac{Y(s)}{R(s)} = \frac{P_1 + P_2\Delta_2 + P_3}{\Delta}$$

Example

Find $\frac{Y(s)}{R(s)}$



Solution

Forward Paths =2

$P_1 = G_1G_2G_3G_4$; $P_2 = G_5G_6G_7G_8$

Feedback loops

$L_1 = G_2H_2$; $L_2 = G_3H_3$; $L_3 = G_6H_6$; $L_4 = G_7H_7$;

Loops L_1 and L_2 do not touch loop L_3 and L_4

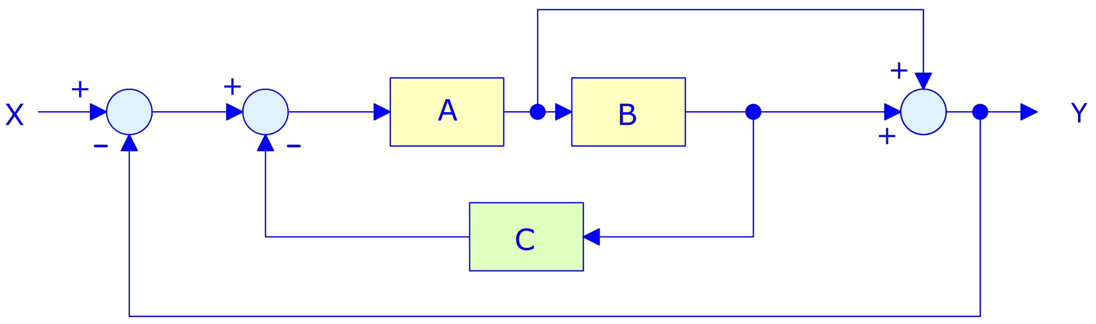
$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)$$

$$\Delta_1 = 1 - (L_3 + L_4) \quad ; \quad \Delta_2 = 1 - (L_1 + L_2)$$

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4)}$$

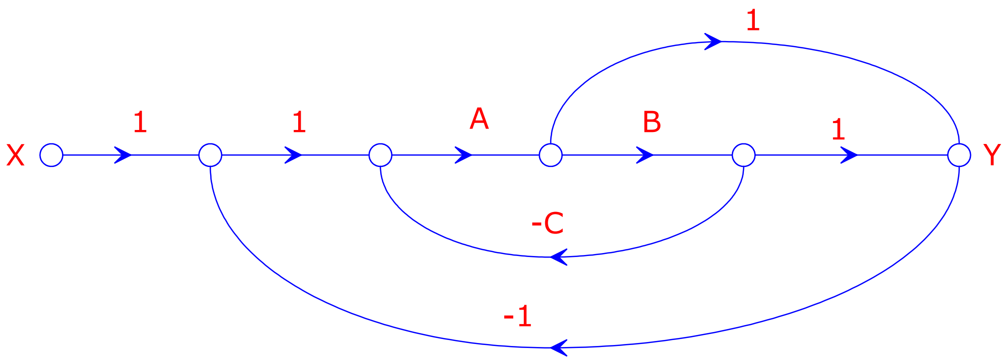
Example

Find $\frac{Y}{X}$ using Mason's rule



Solution

First, draw the SFG



$P_1 = AB \ ; \ \ P_2 = A$

$$\Delta = 1 - (-ABC - AB - A)$$

$$\Delta = 1 + ABC + AB + A$$

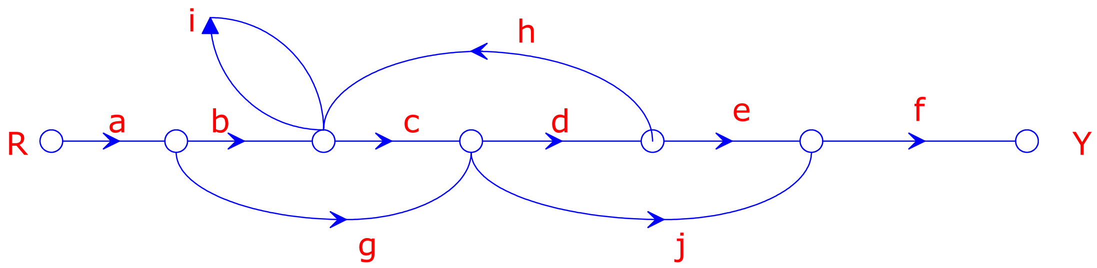
$\Delta_1 = 1 \ ; \ \Delta_2 = 1$

$$\frac{Y}{X} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{A(1+B)}{1+ABC+AB+A}$$

Same answer was obtained in Lecture # 6 using block diagram reduction technique

Example

Find $\frac{Y}{R}$



Solution

$$\Delta = 1 - \{cdh + i\}$$

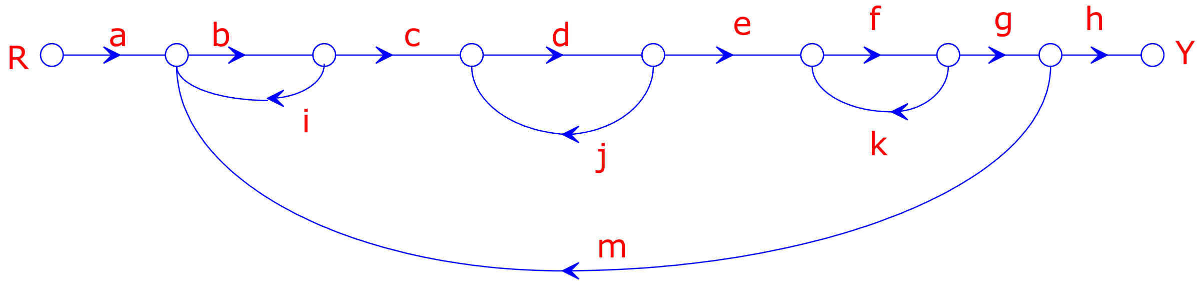
$P_1 = abcdef$; $\Delta_1 = 1$
$P_2 = agdef$; $\Delta_2 = 1 - i$
$P_3 = agjf$; $\Delta_3 = 1 - i$
$P_4 = abcjf$; $\Delta_4 = 1$

$$\frac{Y}{R} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$\frac{Y}{R} = \frac{abcdef + agdef\,(1 - i) + agjf\,(1 - i) + abcjf}{1 - cdh - i}$$

Example

Find $\frac{Y}{R}$



Solution

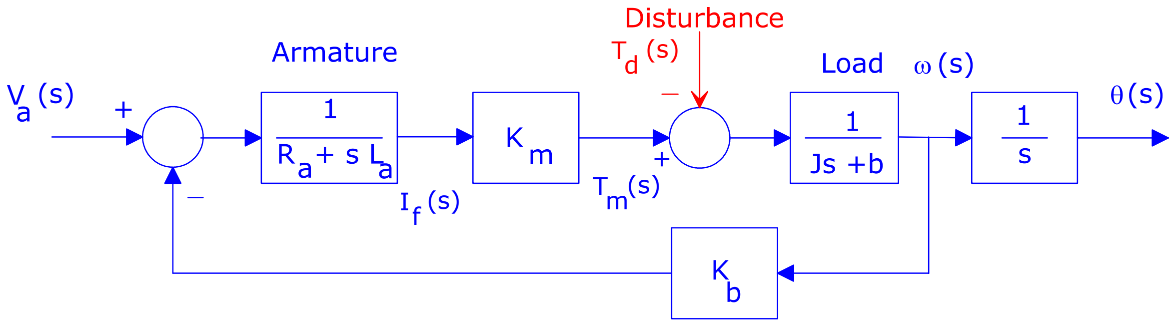
$$\Delta = 1 - \{ib + dj + fk + bcdefgm\} + \{bidj + bifk + djfk\} - \{bidjfk\}$$

$$P_1 = abcdefgh \ ; \ \Delta_1 = 1$$

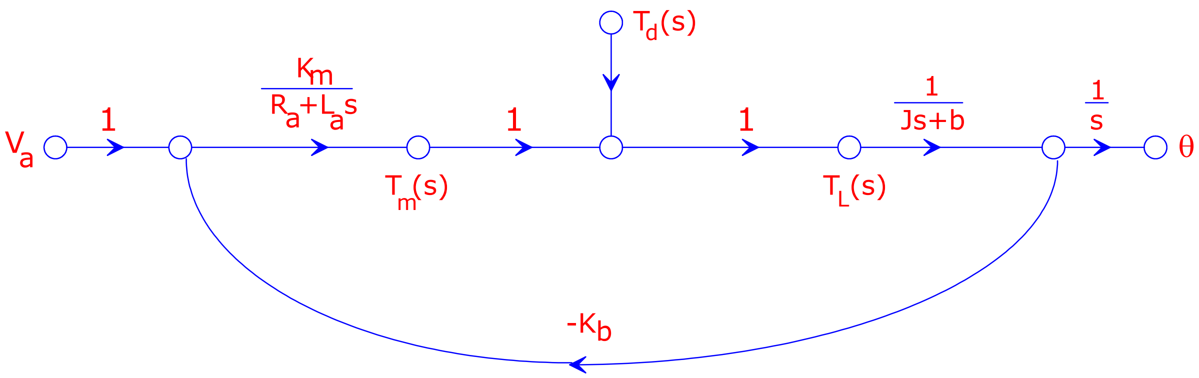
$$\frac{Y}{R} = \frac{P_1 \Delta_1}{\Delta} = \frac{abcdefgh}{1 - \{ib + dj + fk + bcdefgm\} + \{bidj + bifk + djfk\} - \{bidjfk\}}$$

Example

The block diagram of the armature-controlled dc-motor is shown. Find $\frac{\theta(s)}{V_a(s)}$ *using Mason's rule*



Solution



$$P_1(s) = \frac{1}{s}\left(\frac{K_m}{R_a + L_a s}\right)\left(\frac{1}{Js + b}\right);$$

$$L_1(s) = -k_b\left(\frac{K_m}{R_a + L_a s}\right)\left(\frac{1}{Js + b}\right)$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

Same answer as derived before