2. Mathematical Models of Systems

2.1 Introduction

To understand and control complex systems, one must obtain quantitative mathematical models of these systems. The term model, as it is used and understood by control engineers, means a set of differential equations that describe the dynamic behavior of the process. The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process. For mechanical systems, one utilizes Newton's laws, and for electrical systems Kirchhoff's voltage and current laws. Some examples are given to demonstrate how to write the differential equations.

2.2 Differential Equations of Physical Systems

Mechanical systems

The corner stone for obtaining a mathematical model for any mechanical system is Newton's law,

$$\boldsymbol{F} = m\boldsymbol{a}$$

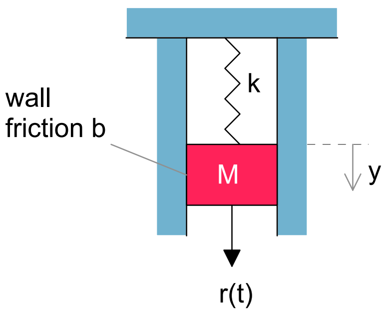
Where

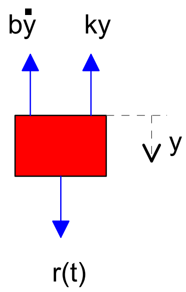
$\boldsymbol{F}$ vector sum of all forces applied to each body in a system, newtons (N) or pounds (lb),

a vector acceleration of each body with respect to an inertial reference frame, m/sec^2 or ft/sec^2,

m mass of the body, kg or slug.

Consider the simple spring-mass-damper shown below. (This system could represent, for example, an automobile shock absorber). In this example, we model the wall friction as a viscous damper, that is, the friction force is linearly proportional to the velocity of the mass.





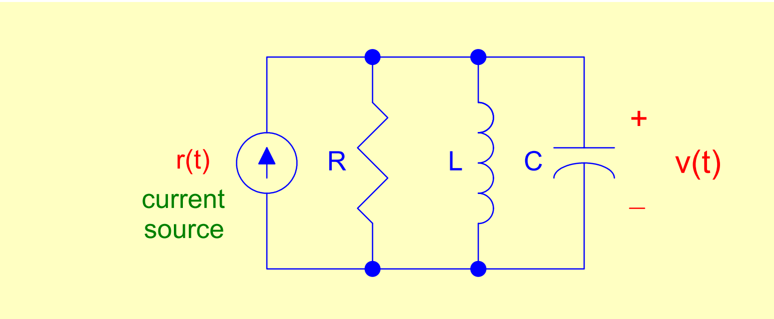
Let k be the spring constant and b the friction coefficient, then summing the forces acting on M and utilizing Newton's second law yields

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

The above equation is a linear constant coefficient differential equation of second order.

Electrical Systems

Consider the RLC circuit shown below.



Using KCL, one obtains the following integrodifferential equation,

$$\frac{v(t)}{R}+C\frac{dv(t)}{dt}+\frac{1}{L}\int v(t)dt=r(t)$$

2.5 The Transfer function of linear systems

The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

Example 1

Find the transfer function of the spring-mass-damper system considered earlier.

Solution

The differential equation of the spring-mass-damper system is given by:

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

The equation can be rewritten with zero initial conditions as follows:

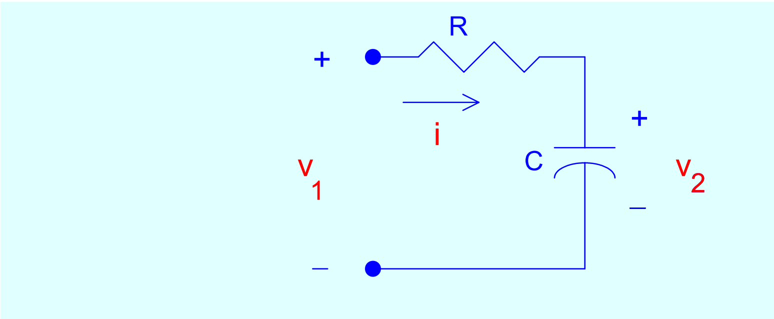
$$Ms^2Y(s)+bsY(s)+kY(s)=R(s)$$

Then the transfer function is :

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$

Example 2

Find the transfer function of the RC circuit shown below.



Solution

The transfer function of the RC network is obtained by writing the KVL equation, yielding

$$V_1(s) = (R + \frac{1}{Cs})I(s) \quad ; \quad V_2(s) = (\frac{1}{Cs})I(s)$$

Then the transfer function is :

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1} = \frac{1}{\tau s + 1}$$

where $\tau = RC$ is the **time constant** of the network. The single pole of $G(s)$ is $s = -\frac{1}{\tau}$.