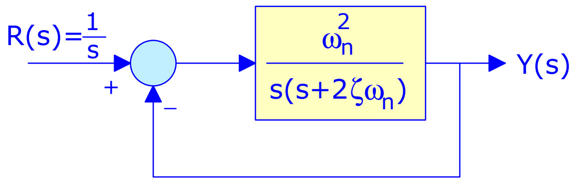
5. THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

EVALUATION OF $T_P, P.O., T_R, T_S$ FOR UNDERDAMPED SECOND-ORDER SYSTEMS

Let us consider a generalized single loop second order system.
The closed-loop output is

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$$



With a unit step input, we obtain

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{\omega_n^2}{s\left(s + \zeta\omega_n \pm j_n\sqrt{1-\zeta^2}\right)}$$

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\, t + \theta)\ ;\ \theta = \cos^{-1}\zeta$$

EVALUATION OF T_p

T_p is found by differentiating $y(t)$ and finding the first zero crossing after $t=0$.This task is simplified by differentiating in the frequency domain. Assuming zero initial conditions, we get

$$\mathcal{L}^{-1}[\dot{y}(t)] = sY(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\mathcal{L}^{-1}[\dot{y}(t)] = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Therefore,

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n\sqrt{1-\zeta^2}\, t$$

Setting the derivative equal to zero yields

$$\omega_n\sqrt{1-\zeta^2}\, t = n\pi\ \ ;\ \ \rightarrow\ \ t = \frac{\pi n}{\omega_n\sqrt{1-\zeta^2}}\quad n = 0, 1, 2, 3, \ldots$$

Each value of n yields the time for local maxima or minima. The first peak, which occurs at the peak time, T_p, is found by Letting $n=1$:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Notice that T_p is a function of ζ and ω_n

EVALUATION OF *P.O.*

The peak response M_{pt} is found by evaluating $y(t)$ at the peak time T_p.

$$M_{pt} = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \sin(\pi+\theta) = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Therefore, The percent overshoot is

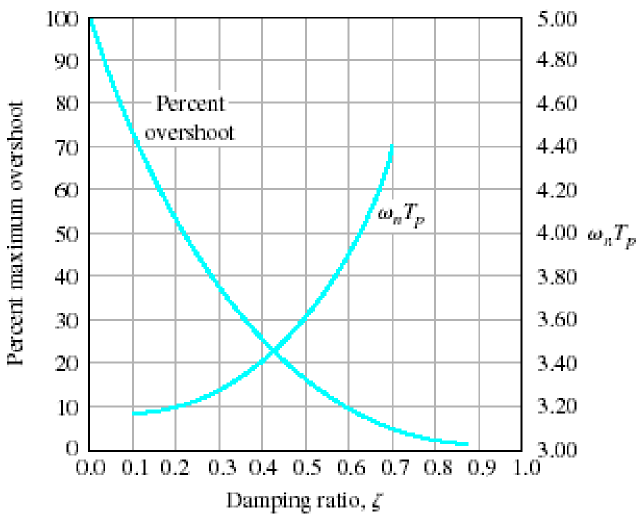
$$P.O. = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \, 100$$

Notice that $P.O.$ is a function only of ζ

The inverse equation allows one to find ζ given $P.O.$

$$\zeta = \frac{-\ln(\frac{P.O.}{100})}{\sqrt{\pi^2 + \ln^2(\frac{P.O.}{100})}}$$

The percent overshoot and the normalized peak time, $\omega_n T_p$, versus the damping ration is shown in the figure.



EVALUATION OF T_s

The settling time is the time required for the response $y(t)$ to reach and stay within a specified absolute percentage δ of the final value. Using a value of $\delta = 2\%$, the settling time occurs when

$$1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_s} = 1.02 \;\rightarrow\; \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_s} = 0.02 \rightarrow T_s = -\frac{\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

the above equation can be approximated to

$e^{-\zeta\omega_n T_s} < 0.02 \;\; \Rightarrow \; \zeta\omega_n T_s = 4$

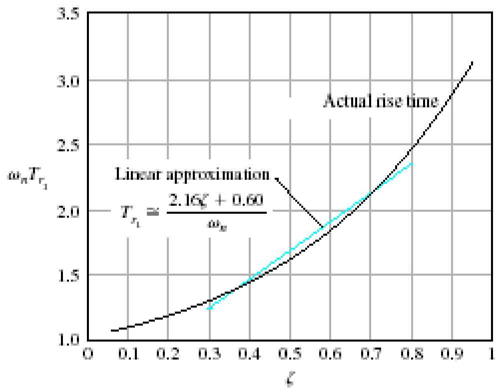
$$T_s = \frac{4}{\zeta\omega_n}$$

Notice that T_s is a function of ζ and ω_n

The settling time for the 2% criterion is approximately four time constants ($\tau = \frac{1}{\zeta\omega_n}$).

Evaluation of T_r

An analytical expression for the rise time is difficult to obtain. However, using a computer, we solve the equation of $y(t)$for the values of $\omega_n t$ that yield $y(t) = 0.1$ and $y(t) = 0.9$. Subtracting the two values of $\omega_n t$ yields the normalized rise time as shown in the figure. We can utilize the linear approximation:



$$T_r = \frac{2.16\zeta + 0.60}{\omega_n}$$

Notice that T_r is a function of ζ and ω_n

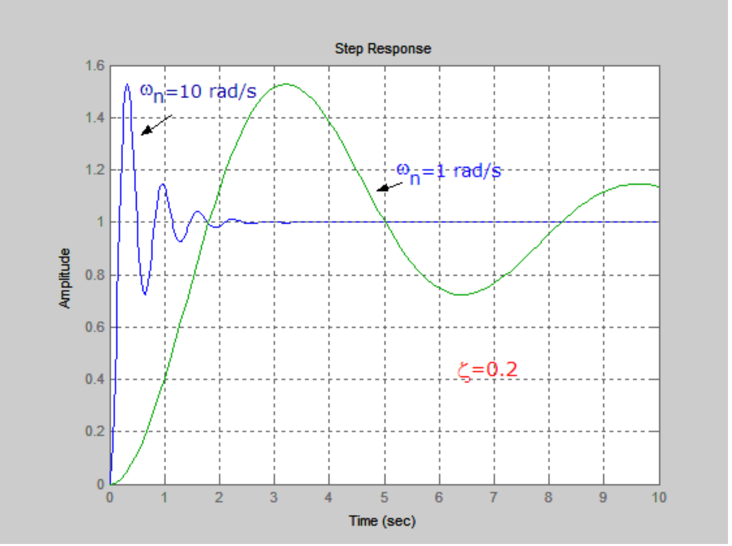
[Accurate for $0.3 \le \zeta \le o.8$]

Effects of ω_n

For a given ζ, the response is faster for larger ω_n, as shown in the figure

Why is the overshoot the same?

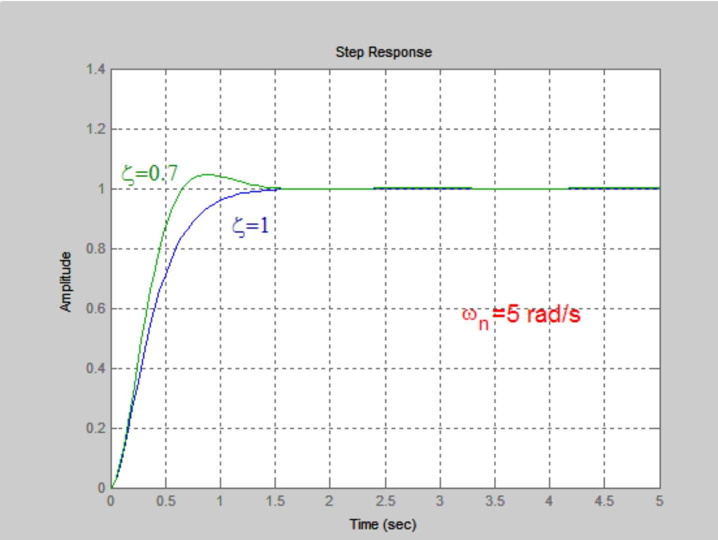
Find T_p, T_r, T_s and $P.O.$ from graphs and from equations. Any comments?



Effects of ζ

For a given ω_n, the response is faster for lower ζ, as shown in the figure.

Find T_p, T_r, T_s and $P.O.$ from graphs and from equations. Any comments?



Skill-Assessment Exercise

Find $\zeta, \omega_n, T_p, T_r, T_s,$ and $P.O.$ for a system whose transfer function is

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

Answers:
$\zeta = 0.421, \omega_n = 19\,\text{rad/s}, T_p = 0.182\,\text{s}, T_r = 0.079\,s, T_s = 0 = .5\,s, P.O. = 23\%$

Plot the step response using MATLAB and check the accuracy of the results.

Drill Problem [Due on Wednesday 30 October, 2002]

Consider the servo system shown.

1. Design the system to have the fastest response without overshoot, and a settling time of 0.25 s.
2. Simulate the system to verify the design.