3. STATE VARIABLE MODELS (cont.)

SOLUTION OF THE STATE DIFFERENTIAL EQUATION

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first-order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu$$

Taking the Laplace transform, we have

$$sX(s) - x(0) = aX(s) + bU(s) \;\rightarrow\; X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

The inverse Laplace transform results in

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

By analogy, the solution of the state differential equation $\dot{\mathbf{x}} = a\mathbf{x} + b\mathbf{u}$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau \,,$$ where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$ (converges for all finite t and A)

Also, the Laplace transform of the state differential equation $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is

$$\mathbf{X}(s) = [sI - A]^{-1}\mathbf{x}(0) + [sI - A]^{-1}B\mathbf{U}(s)$$

Now, we note that

$$e^{At} = \mathcal{L}^{-1}\,[sI - A]^{-1}$$

$[sI - A]^{-1} \triangleq \Phi(s)$ is known as the fundamental or state transition matrix

therefore

$e^{At} = \Phi(t)$; $\Phi(t) = \mathcal{L}^{-1}\,\Phi(s)$

The solution of the state differential equation $\dot{\mathbf{x}} = a\mathbf{x} + b\mathbf{u}$ is then

$$\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\,B\,\mathbf{u}(\tau)d\tau$$