Solution

Prob 1:

Ans:

$$y(s)=\frac{\omega_n^2 p}{s(s+p)(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{C_1}{s}+\frac{C_2}{s+p}+\frac{C_3}{s+\zeta\omega_n-j\omega_d}+\frac{C_4}{s+\zeta\omega_n+j\omega_d}$$

$$C_1=1,\ C_2=\frac{-\omega_n^2}{p^2+\omega_n^2-2\zeta\omega_n p},\ C_3=\frac{p}{2\sqrt{(1-\zeta^2)(p^2+\omega_n^2-2\zeta\omega_n p)}}e^{j\theta}=Ke^{j\theta}$$

$$\theta=\operatorname{atan}\frac{\sqrt{1-\zeta^2}}{\zeta}+\operatorname{atan}\frac{-\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n}-90^o$$

$$C_4=C_3^*=Ke^{-j\theta}$$

$$\begin{aligned}y(t)&=1+C_2e^{-pt}+Ke^{-\zeta\omega_n t}\left[e^{j(\omega_d t+\theta)}+e^{-j(\omega_d t+\theta)}\right]\\&=1+C_2e^{-pt}+2Ke^{-\zeta\omega_n t}\cos(\omega_d t+\theta)\\&=1+C_2e^{-pt}+2Ke^{-\zeta\omega_n t}\sin(\omega_d t-\gamma)\\&=1-\underbrace{\frac{\omega_n^2}{p^2+\omega_n^2-2\zeta\omega_n p}}_{A}e^{-pt}+\underbrace{\frac{p}{\sqrt{(1-\zeta^2)(p^2+\omega_n^2-2\zeta\omega_n p)}}}_{B}e^{-\zeta\omega_n t}\sin(\omega_d t-\gamma)\end{aligned}$$

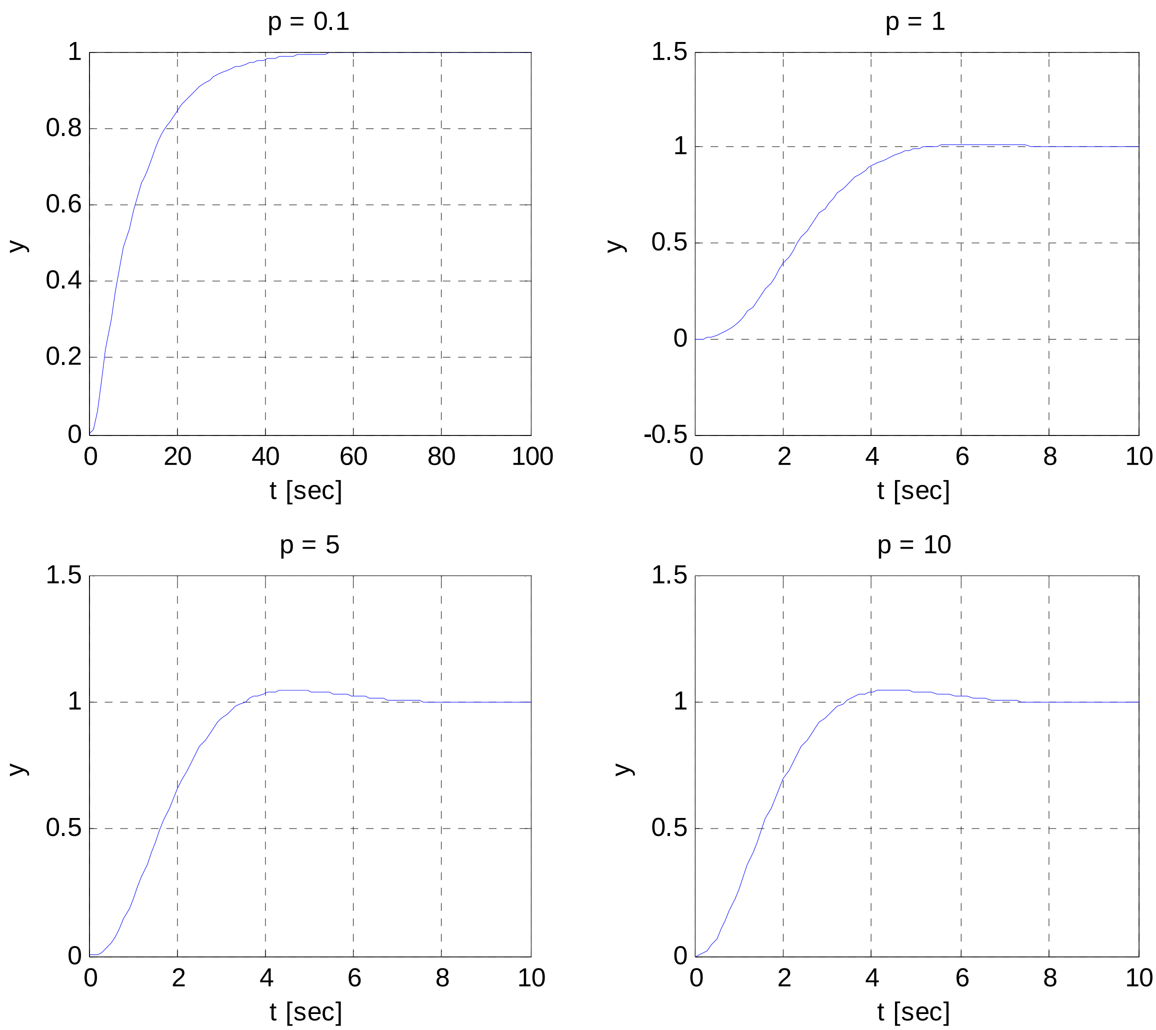
$$\omega_d=\omega_n\sqrt{1-\zeta^2},\ \gamma=\operatorname{atan}\frac{\sqrt{1-\zeta^2}}{-\zeta}+\operatorname{atan}\frac{\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n}$$

Note: when computing the arc-tangent values numerically, use **atan2(**y,x**)** function.

a) When $p>>\omega_n$, $e^{-\zeta\omega_n t}\sin(\omega_d t-\gamma)$ becomes the dominant term ($A\cong 0$; $B\cong 1$).

b) When $p<<\omega_n$, e^{-pt} becomes dominant ($A\cong -1$; $B\cong 0$).

d) When $p>5\omega_n$, the effect of the additional pole at $-p$ becomes less significant.




```
% m file to solve Problem 1

clear;

wn = 1;                    % [rad/sec]
zeta = 0.7;                % [ ]
wd = wn*sqrt(1-zeta^2); % [rad/sec]

p = 0.1;                   % [rad/sec]
B = -wn^2/(p^2 + wn^2 - 2*zeta*wn*p);
C = p / sqrt((1-zeta^2)*(p^2 + wn^2 - 2*zeta*wn*p));

% be sure to use atan2 instead of atan in computing the phase %
gamma = atan2(sqrt(1-zeta^2),-zeta) + atan2(wn*sqrt(1-zeta^2),(p-zeta*wn));
t1 = linspace(0,10/min(p,wn),100);

y1 = 1 + B*exp(-p*t1) + C*exp(-zeta*wn*t1).*sin(wd*t1-gamma);

G = tf([p*wn^2],conv([1 p],[1 2*zeta*wn wn^2]));
[y t] = step(G,max(t1));

figure(1); clf; zoom on;
plot(t,y,'b',t1,y1,'r--');
title('Verification of Analytical Solution');
ylabel('y'); xlabel('t [sec]'); legend('Numerical','Analytical');
```

```
p = 0.1;   t1 = linspace(0,10/min(p,wn),100);
B = -wn^2/(p^2 + wn^2 - 2*zeta*wn*p);
C = p / sqrt((1-zeta^2)*(p^2 + wn^2 - 2*zeta*wn*p));
gamma = atan2(sqrt(1-zeta^2),-zeta) + atan2(wn*sqrt(1-zeta^2),(p-zeta*wn));
y1 = 1 + B*exp(-p*t1) + C*exp(-zeta*wn*t1).*sin(wd*t1-gamma);

p = 1.0;   t2 = linspace(0,10/min(p,wn),100);
B = -wn^2/(p^2 + wn^2 - 2*zeta*wn*p);
C = p / sqrt((1-zeta^2)*(p^2 + wn^2 - 2*zeta*wn*p));
gamma = atan2(sqrt(1-zeta^2),-zeta) + atan2(wn*sqrt(1-zeta^2),(p-zeta*wn));
y2 = 1 + B*exp(-p*t2) + C*exp(-zeta*wn*t2).*sin(wd*t2-gamma);

p = 5.0;  t3 = linspace(0,10/min(p,wn),100);
B = -wn^2/(p^2 + wn^2 - 2*zeta*wn*p);
C = p / sqrt((1-zeta^2)*(p^2 + wn^2 - 2*zeta*wn*p));
gamma = atan2(sqrt(1-zeta^2),-zeta) + atan2(wn*sqrt(1-zeta^2),(p-zeta*wn));
y3 = 1 + B*exp(-p*t3) + C*exp(-zeta*wn*t3).*sin(wd*t3-gamma);

p = 10.0; t4 = linspace(0,10/min(p,wn),100);
B = -wn^2/(p^2 + wn^2 - 2*zeta*wn*p);
C = p / sqrt((1-zeta^2)*(p^2 + wn^2 - 2*zeta*wn*p));
gamma = atan2(sqrt(1-zeta^2),-zeta) + atan2(wn*sqrt(1-zeta^2),(p-zeta*wn));
y4 = 1 + B*exp(-p*t4) + C*exp(-zeta*wn*t4).*sin(wd*t4-gamma);

figure(2); clf; zoom on;
subplot(2,2,1); plot(t1,y1); ylabel('y'); xlabel('t [sec]'); title(['p = '
num2str(0.1)]); grid;
subplot(2,2,2); plot(t2,y2); ylabel('y'); xlabel('t [sec]'); title(['p = '
num2str(1.0)]); grid;
subplot(2,2,3); plot(t3,y3); ylabel('y'); xlabel('t [sec]'); title(['p = '
num2str(5.0)]); grid;
subplot(2,2,4); plot(t4,y4); ylabel('y'); xlabel('t [sec]'); title(['p = '
num2str(10.0)]); grid;
```

Solution

Prob 2

Ans:

a) Characteristic polynomial: $s^4 + 2s^3 + 3s^2 + 8s + 8 = 0$

Routh's table:

Row					
4	s^4	1	3	8	0
3	s^3	2	8	0	0
2	s^2	-1	8	0	
1	s^1	24	0		
0	s^0	8	0		

Two sign changes → two RHP poles. The system is unstable.

b) Characteristic polynomial: $s^3 + s^2 + 2s + 8 = 0$

Routh's table:

Row				
3	s^3	1	2	0
2	s^2	1	8	0
1	s^1	-6	0	
0	s^0	8		

Two sign changes → two RHP poles. The system is unstable.

c) Characteristic polynomial: $s^5 + 2s^4 + +3s^3 + 7s^2 + 4s + 4 = 0$

Routh's table:

Row					
5	s^5	1	3	4	0
4	s^4	2	7	4	0
3	s^3	-1/2	2	0	0
2	s^2	15	4	0	
1	s^1	32/15	0		
0	s^0	4	0		

Two sign changes → two RHP poles. The system is unstable.

Solution

Prob 3:

Ans:

a) Characteristic polynomial: $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$

Routh's table:

Row					
4	s^4	1	32	100	0
3	s^3	8	80	0	0
2	s^2	22	100	0	
1	s^1	$\frac{480}{11}$	0		
0	s^0	100	0		

No sign changes → No RHP roots.

b) Characteristic polynomial: $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$

Routh's table:

Row					
5	s^5	1	30	344	0
4	s^4	10	80	480	0
3	s^3	22	296	0	0
2	s^2	$\frac{-600}{11}$	480	0	
1	s^1	$\frac{2448}{5}$	0		
0	s^0	480	0		

Two sign changes → two RHP roots.

c) Characteristic polynomial: $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$ (Necessary condition not met, indicating RHP roots). Routh's table:

Row					
4	s^4	1	7	8	0
3	s^3	2	-2	0	0
2	s^2	8	8	0	
1	s^1	-4	0		
0	s^0	8	0		

Two sign changes → two RHP roots.

d) Characteristic polynomial: $s^3+s^2+20s+78=0$

Row				
3	s^3	1	20	0
2	s^2	1	78	0
1	s^1	-58	0	
0	s^0	78		

Two sign changes → two RHP roots.

e) Characteristic polynomial: $s^4+6s^2+25=0$

Routh's table:

Row					
4	s^4	1	6	25	
3	s^3	0	0	0	(Special Case II)
new 3	s^3	4	12	0	Form auxiliary polynomial using 4th row:
2	s^2	3	25	0	$P(s)=s^4+6s^2+25$ and take its derivative to construct new 3rd row:
1	s^1	$-\frac{64}{3}$	0		$dP/ds=4s^3+12s$
0	s^0	25	0		

Two sign changes → two RHP roots.

Location of roots which cause 3rd row to vanish:

$$s^4+6s^2+25=0 \rightarrow s^2=-3\pm j4=5e^{j126.90°} \rightarrow s_{1,2}=\pm\sqrt{5}e^{j\frac{126.87°}{2}}=1\pm j2,-1\pm j2$$

Note that these roots are symmetrical with respect to the imaginary axis.

The above result can also be verified in Matlab by executing: `roots([1 0 6 0 25]).`

Solution

Prob:

Characteristic polynomial: $s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$

Routh's table:

Row					
5	s^5	1	10	5	0
4	s^4	5	10	K	0
3	s^3	8	$\frac{25-K}{5}$	0	
2	s^2	$\frac{55+K}{8}$	K	0	
1	s^1	$\frac{-K^2-350K+1375}{5K+275}$	0		
0	s^0	K	0		

Necessary condition for stability: all coefficients should be greater than zero, Hence K > 0 (1)

Stability conditions from the Routh array:

$$\frac{55+K}{8} > 0 \Rightarrow K > -55 \quad (2)$$

$$5K + 275 > 0 \Rightarrow K > -55 \quad (3)$$

$$-K^2 - 350K + 1375 > 0 \Rightarrow -353.98 < K < 3.89 \quad (4)$$

Combining (1), (2), (3) and (4) gives $0 < K < 3.89$.

Solution

Problem 5:

a) $y(s) = \frac{KK_p}{s^2 + as + KK_p} r(s) - \frac{K}{s^2 + as + KK_p} d(s)$.

b) $e = r - y = \frac{s^2 + as}{s^2 + as + KK_p} r(s) + \frac{K}{s^2 + as + KK_p} d(s)$.

c) $e_{ss} = \lim s.e(s) = \frac{2\zeta}{\omega_n} A + \frac{K}{\omega_n^2} D$.

As the natural frequency ω_n (~ bandwidth) increases, the steady state error e_{ss} decreases.