EE 380

SOLUTION HW 3

E3.5 From the block diagram we determine that the state equations are

$$\dot{x_2} = -(fk+d)x_1 + ax_1 + fu$$
$$\dot{x_1} = -kx_2 + u$$

and the output equation is

$$y = bx_2 \ .$$

Therefore,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u \ ,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -k \\ a & -(fk+d) \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 1 \\ f \end{bmatrix} , \quad \mathbf{C} = \begin{bmatrix} 0 & b \end{bmatrix} \text{ and } \quad \mathbf{D} = [0] \ .$$

E3.7 The state equations are

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -100x_1 - 20x_2 + u$$

or, in matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ .$$

So, the characteristic equation is determined to be

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -1 \\ 100 & \lambda + 20 \end{bmatrix} = \lambda^2 + 20\lambda + 100 = (\lambda + 10)^2 = 0 \ .$$

Thus, the roots of the characteristic equation are

$$\lambda_1 = \lambda_2 = -10 \ .$$

E3.8 The characteristic equation is

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 5 & \lambda+2 \end{bmatrix} = \lambda(\lambda^2+2\lambda+5) = 0 \ .$$

Thus, the roots of the characteristic equation are

$$\lambda_1 = 0 \ , \quad \lambda_2 = -1 + j2 \text{ and } \quad \lambda_3 = -1 - j2 \ .$$

E3.11 A state variable representation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}r$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \ , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \quad \mathbf{C} = \begin{bmatrix} 12 & 4 \end{bmatrix} \ .$$

E3.15 The equations of motion are

$$m\ddot{x} + kx + k_1(x-q) + b\dot{x} = 0$$
$$m\ddot{q} + kq + b\dot{q} + k_1(q-x) = 0 \ .$$

In state variable form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k+k_1)}{m} & -\frac{b}{m} & \frac{k_1}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & 0 & -\frac{(k+k_1)}{m} & -\frac{b}{m} \end{bmatrix} \mathbf{x}$$

where $x_1 = x, x_2 = \dot{x}, x_3 = q$ and $x_4 = \dot{q}$.

E3.19 First, compute the matrix

$$sI - \mathbf{A} = \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix} .$$

Then, $\Phi(s)$ is

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}$$

where $\Delta(s) = s^2 + 4s + 3$, and

$$G(s) = \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+4}{\Delta(s)} & \frac{1}{\Delta(s)} \\ -\frac{3}{\Delta(s)} & \frac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{10}{s^2 + 4s + 3} .$$

P3.5 (a) The closed-loop transfer function is

$$T(s) = \frac{s+1}{s^3 + 5s^2 - 5s + 1} .$$

(b) A matrix differential equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 5 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} .$$

The block diagram is shown in Figure P3.5.

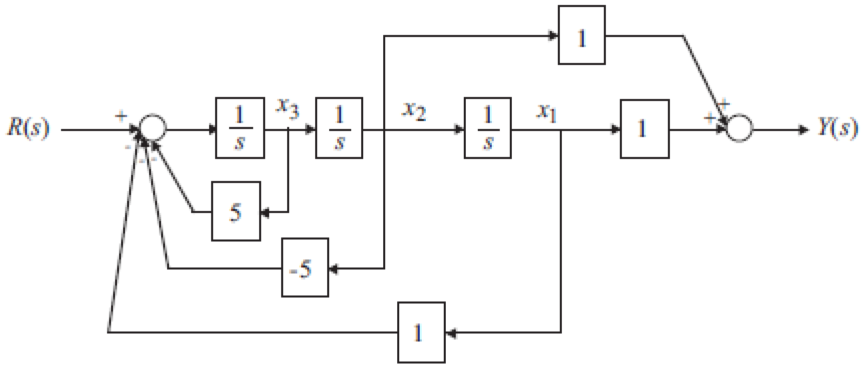


FIGURE P3.5
Block diagram model.

P3.10 (a) From the signal flow diagram, we determine that a state-space model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} K_1 & -K_2 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \, .$$

(b) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + (K_2 + K_1)s + 2K_1K_2 = 0 \, .$$

(c) When $K_1 = K_2 = 1$, then

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} .$$

The state transition matrix associated with $\mathbf{A}$ is

$$\Phi = \mathcal{L}^{-1} \left\{ [s\mathbf{I} - \mathbf{A}]^{-1} \right\} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} .$$

P3.12 (a) The phase variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [40 \;\; 8 \;\; 0]\mathbf{x} \, .$$

(b) The canonical representation is

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \mathbf{z} + \begin{bmatrix} -0.5728 \\ 4.1307 \\ 4.5638 \end{bmatrix} r$$

$$y = [-5.2372 \quad -0.4842 \quad -0.2191]\mathbf{z}$$

(c) The state transition matrix is

$$\Phi(t) = \left[\; \Phi_1(t) \vdots \Phi_2(t) \vdots \Phi_3(t) \; \right] ,$$

where

$$\Phi_1(t) = \begin{bmatrix} e^{-6t} - 3e^{-4t} + 3e^{-2t} \\ -6e^{-6t} + 12e^{-4t} - 6e^{-2t} \\ 36e^{-6t} - 48e^{-4t} + 12e^{-2t} \end{bmatrix} \quad \Phi_2(t) = \begin{bmatrix} \frac{3}{4}e^{-6t} - 2e^{-4t} + \frac{5}{4}e^{-2t} \\ -\frac{9}{2}e^{-6t} + 8e^{-4t} - \frac{5}{2}e^{-2t} \\ 27e^{-6t} - 32e^{-4t} + 5e^{-2t} \end{bmatrix}$$

$$\Phi_3(t) = \begin{bmatrix} \frac{1}{8}e^{-6t} - \frac{1}{4}e^{-4t} + \frac{1}{8}e^{-2t} \\ -\frac{3}{4}e^{-6t} + e^{-4t} - \frac{1}{4}e^{-2t} \\ \frac{9}{2}e^{-6t} - 4e^{-4t} + \frac{1}{2}e^{-2t} \end{bmatrix} .$$

P3.19 Define the state variables as

$$x_1 = \phi_1 - \phi_2$$
$$x_2 = \frac{\omega_1}{\omega_o}$$
$$x_3 = \frac{\omega_2}{\omega_o} .$$

Then, the state equations of the robot are

$$\dot{x}_1 = \omega_o x_2 - \omega_o x_3$$
$$\dot{x}_2 = \frac{-J_2\omega_o}{J_1 + J_2} x_1 - \frac{b}{J_1} x_2 + \frac{b}{J_1} x_3 + \frac{K_m}{J_1\omega_o} i$$
$$\dot{x}_3 = \frac{J_1\omega_o}{J_1 + J_2} x_2 + \frac{b}{J_2} x_2 - \frac{b}{J_2} x_3$$

or, in matrix form

$$\dot{\mathbf{x}} = \omega_o \begin{bmatrix} 0 & 1 & -1 \\ a-1 & -b_1 & b_1 \\ a & b_2 & -b_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix} i$$

where

$$a = \frac{J_1}{(J_1 + J_2)} , \quad b_1 = \frac{b}{J_1\omega_o} , \quad b_2 = \frac{b}{J_2\omega_o} \text{ and } \quad d = \frac{K_m}{J_1\omega_o} .$$

P3.24 (a) The phase variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -30 & -31 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = [1 \ \ 0 \ \ 0]\mathbf{x} .$$

(b) The input feedforward representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 & 0 \\ -31 & 0 & 1 \\ -30 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$
$$y = [1 \ \ 0 \ \ 0]\mathbf{x} .$$

(c) The physical variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [1 \;\; 0 \;\; 0]\mathbf{x} \; .$$

(d) The decoupled representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \mathbf{x} \; .$$

DP3.2 The desired transfer function is

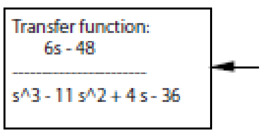
$$\frac{Y(s)}{U(s)} = \frac{10}{s^2 + 4s + 3} \; .$$

The transfer function derived from the phase variable representation is

$$\frac{Y(s)}{U(s)} = \frac{d}{s^2 + bs + a} \; .$$

Therefore, we select $d = 10$, $a = 3$ and $b = 4$.

CP3.2 The m-file script to compute the transfer function models using the **tf** function is shown in Figure CP3.2.




```
% Part (b)
A=[1 1 0;-2 0 4;6 2 10]; B=[-1;0;1]; C=[0 1 0]; D=[0];
sys_ss=ss(A,B,C,D);
sys_tf = tf(sys_ss)
```

CP3.7 The m-file script and system response is shown in Figure CP3.7.

```
a=[0 1;-2 -3]; b=[0;1]; c=[1 0]; d=[0];
sys = ss(a,b,c,d);
x0=[1;0];
t=[0:0.1:10]; u=0*t;
[y,t,x]=lsim(sys,u,t,x0);
plot(t,x(:,1),t,x(:,2),'--')
xlabel('Time (sec)')
ylabel('State Response')
legend('x1','x2',-1)
grid
```

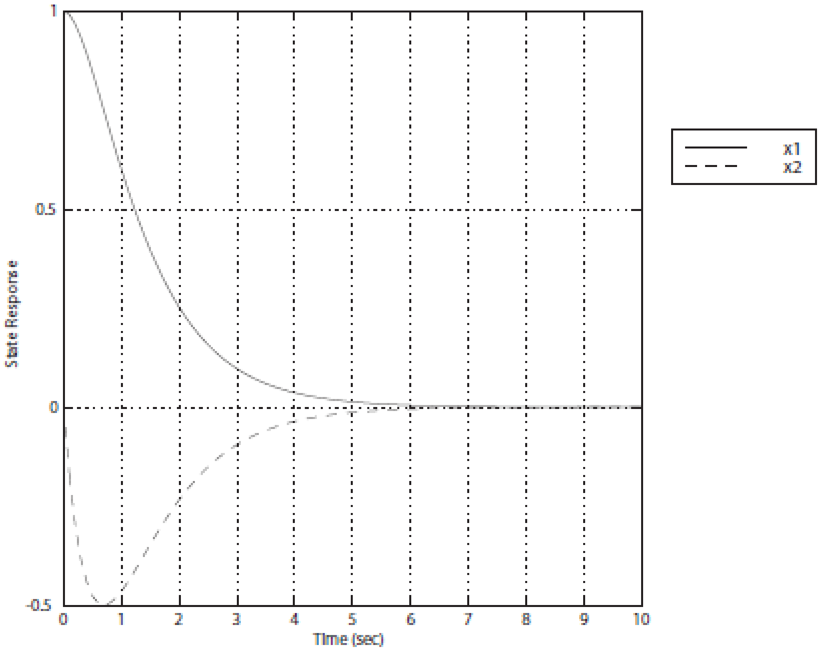


FIGURE CP3.7
Using the **lsim** function to compute the zero input response.