where, in British units,

J = inertia (oz-in.-sec^2) L = screw lead (in.)
W = weight (oz) g = gravitational force (386.4 in./sec^2)

4-3-4 Gear Trains, Levers, and Timing Belts

A gear train, lever, or timing belt over a pulley is a mechanical device that transmits energy from one part of the system to another in such a way that force, torque, speed, and displacement may be altered. These devices can also be regarded as matching devices used to attain maximum power transfer. Two gears are shown coupled together in Fig. 4-12. The inertia and friction of the gears are neglected in the ideal case considered.

The relationships between the torques, T_1 and T_2, angular displacement θ_1 and θ_2, and the teeth numbers N_1 and N_2 of the gear train are derived from the following facts:

1. *The number of teeth on the surface of the gears is proportional to the radii r_1 and r_2 of the gears; that is,*

$$r_1 N_2 = r_2 N_1 \tag{4-32}$$

2. *The distance traveled along the surface of each gear is the same.* Thus

$$\theta_1 r_1 = \theta_2 r_2 \tag{4-33}$$

3. *The work done by one gear is equal to that of the other since there are assumed to be no losses.* Thus

$$T_1 \theta_1 = T_2 \theta_2 \tag{4-34}$$

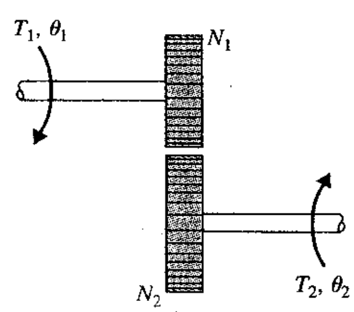

Figure 4-12 Gear train.

for Rotational Motion:
$\sum$ Torques $= J\alpha = J\dfrac{d^2\theta}{dt^2}$

J = inertia, α = angular acceleration
θ = angular displacement

If the angular velocities of the two gears, ω_1 and ω_2, are brought into the picture, Eqs. (4-32) through (4-34) lead to

$$\frac{T_1}{T_2} = \frac{\theta_1}{\theta_2} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} \qquad (4\text{-}35)$$

In practice, gears do have inertia and friction between the coupled gear teeth that often cannot be neglected. An equivalent representation of a gear train with viscous friction, Coulomb friction, and inertia, considered as lumped parameters, is shown in Fig. 4-13, where T denotes the applied torque, T_1 and T_2 are the transmitted torques, F_{c1}, and F_{c2}, are the Coulomb friction coefficients, and B_1 and B_2 are the viscous friction coefficients. The torque equation for gear 2 is

$$T_2(t) = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + F_{c2} \frac{\omega_2}{|\omega_2|} \qquad (4\text{-}36)$$

The torque equation on the side of gear 1 is

$$T(t) = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + F_{c1} \frac{\omega_1}{|\omega_1|} + T_1(t) \qquad (4\text{-}37)$$

By use of Eq. (4-35), Eq. (4-36) is converted to

$$T_1(t) = \frac{N_1}{N_2} T_2(t) = \left(\frac{N_1}{N_2}\right)^2 J_2 \frac{d^2\theta_1(t)}{dt^2} + \left(\frac{N_1}{N_2}\right)^2 B_2 \frac{d\theta_1(t)}{dt} + \frac{N_1}{N_2} F_{c2} \frac{\omega_2}{|\omega_2|} \qquad (4\text{-}38)$$

Equation (4-38) indicates that *it is possible to reflect inertia, friction, compliance, torque, speed, and displacement from one side of a gear train to the other.* The following

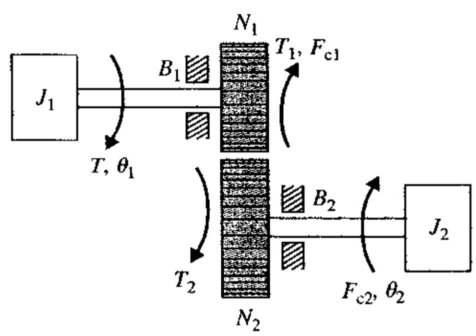

Figure 4-13 Gear train with friction and inertia.

quantities are obtained when reflecting from gear 2 to gear 1:

Inertia: $\left(\frac{N_1}{N_2}\right)^2 J_2$

Angular displacement: $\frac{N_1}{N_2}\theta_2$

Viscous friction coefficient: $\left(\frac{N_1}{N_2}\right)^2 B_2$

Angular velocity: $\frac{N_1}{N_2}\omega_2$

Torque: $\frac{N_1}{N_2}T_2$

Coulomb friction torque: $\frac{N_1}{N_2}F_{c2}\frac{\omega_2}{|\omega_2|}$

Similarly, gear parameters and variables can be reflected from gear 1 to gear 2 simply by interchanging the subscripts in the expressions above.

If a torsional spring effect is present, the spring constant is also multiplied by $(N_1/N_2)^2$ in reflecting from gear 2 to gear 1. Now substituting Eq. (4-38) into Eq. (4-37), we get

$$T(t) = J_{1e}\frac{d^2\theta_1(t)}{dt^2} + B_{1e}\frac{d\theta_1(t)}{dt} + T_F \tag{4-39}$$

where

$$J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 \tag{4-40}$$

$$B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2 \tag{4-41}$$

$$T_F = F_{c1}\frac{\omega_1}{|\omega_1|} + \frac{N_1}{N_2}F_{c2}\frac{\omega_2}{|\omega_2|} \tag{4-42}$$

Example 4-3

Given a load that has inertia 0.05 oz-in.-sec^2 and Coulomb friction torque 2 oz-in., find the inertia and frictional torque reflected through a 1 : 5 gear train ($N_1/N_2 = 1/5$, with N_2 on the load side). The reflected inertia on the side of N_1 is $(\frac{1}{5})^2 \times 0.05 = 0.002$ oz-in.-sec^2. The reflected Coulomb friction is $\frac{1}{5} \times 2 = 0.4$ oz-in. ▲

Timing belts and chain drives serve the same purpose as the gear train except that they allow the transfer of energy over a longer distance without using an excessive number of gears. Figure 4-14 shows the diagram of a belt or chain drive between two pulleys. Assuming that there is no slippage between the belt and the pulleys, it is easy of the state variables and the input force $f(t)$, we have:

$$\text{Force on mass:} \quad M\frac{dv(t)}{dt} = -Bv(t) - f_k(t) + f(t) \tag{4-48}$$

$$\text{Velocity of spring:} \quad \frac{1}{K}\frac{df_k(t)}{dt} = v(t) \tag{4-49}$$

The state equations are obtained by dividing both sides of Eq. (4-48) by M and multiplying Eq. (4-49) by K.

▲ The state variables and state equations of a dynamic system are not unique.

This simple example illustrates that the state equation and state variables of a dynamic system are not unique. The transfer function between $Y(s)$ and $F(s)$ is obtained by taking the Laplace transform on both sides of Eq. (4-44) with zero initial conditions:

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \tag{4-50}$$

The same result is obtained by applying the gain formula to Fig. 4-18(c). ▲

Example 4-5

As another example of writing the dynamic equations of a mechanical system with translational motion, consider the system shown in Fig. 4-19(a). Since the spring is deformed when it is subject to a force $f(t)$, two displacements, y_1 and y_2, must be assigned to the endpoints of the spring. The free-body diagrams of the system are shown in Fig. 4-19(b). The force equations are

$$f(t) = K[y_1(t) - y_2(t)] \tag{4-51}$$

$$K[y_1(t) - y_2(t)] = M\frac{d^2y_2(t)}{dt^2} + B\frac{dy_2(t)}{dt} \tag{4-52}$$

These equations are rearranged as

$$y_1(t) = y_2(t) + \frac{1}{K}f(t) \tag{4-53}$$

$$\frac{d^2y_2(t)}{dt^2} = -\frac{B}{M}\frac{dy_2(t)}{dt} + \frac{K}{M}[y_1(t) - y_2(t)] \tag{4-54}$$

By using the last two equations, the state diagram of the system is drawn in Fig. 4-19(c). The state variables are defined as $x_1(t) = y_2(t)$ and $x_2(t) = dy_2(t)/dt$. The state equations are written directly from the state diagram:

$$\frac{dx_1(t)}{dt} = x_2(t) \tag{4-55}$$

$$\frac{dx_2(t)}{dt} = -\frac{B}{M}x_2(t) + \frac{1}{M}f(t) \tag{4-56}$$

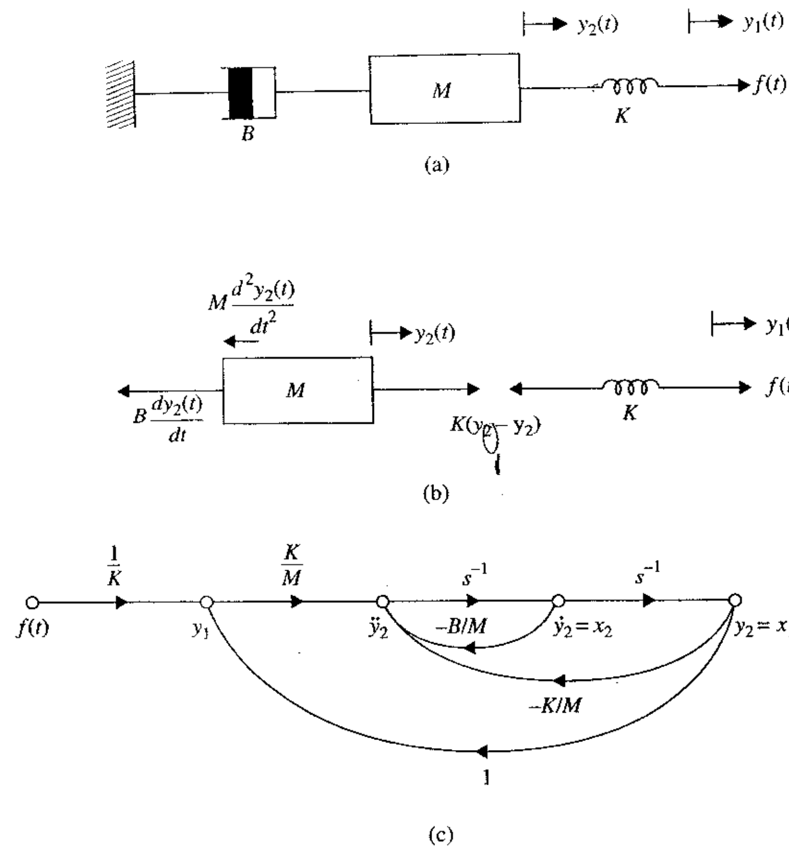

Figure 4-19 Mechanical system for Example 4-5. (a) Mass–spring–friction system. (b) Free-body diagram. (c) State diagram.

As an alternative, we can assign the velocity $v(t)$ of the mass M as one state variable and the force $f_k(t)$ on the spring as the other state variable. We have

$$\frac{dv(t)}{dt} = -\frac{B}{M}v(t) + \frac{1}{M}f_k(t) \tag{4-57}$$

$$f_k(t) = f(t) \tag{4-58}$$

One may wonder why there is only one state equation in Eq. (4-48), whereas there are two state variables in $v(t)$ and $f_k(t)$. The two state equations of Eqs. (4-55) and (4-56) clearly show that the system is of the second order. The situation is better explained by referring to the analogous electric network of the system shown in Fig. 4-20. Although the network has two energy-storage elements in L and C, and thus there should be two state variables, the voltage across the capacitance, $e_c(t)$, in this case is redundant, since it is equal to the applied voltage $e(t)$. Equations (4-57) and (4-56) can provide only the solutions to the velocity of M, $v(t)$, which