From the second condition $\lambda = 1$. Substituting in the first condition

$$2x + (1)(2x - 6) = 0 \quad \text{or} \quad x = 1.5$$

Substituting for x in the third condition results in $y = 3.20156$. Thus the minimum value of the function is

$$f(\hat{x}, \hat{y}) = (1.5)^2 + (3.20156)^2 = 12.5$$

7.6. Find the minimum value of the function

$$f(x, y) = x^2 + y^2$$

subject to one equality constraint

$$g(x, y) = x^2 - 5x - y^2 + 20 = 0$$

and one inequality constraint

$$u(x, y) = 2x + y \geq 6$$

The unconstrained cost function from (7.16) is

$$\mathcal{L} = x^2 + y^2 + \lambda(x^2 - 5x - y^2 + 20) + \mu(2x + y - 6)$$

The resulting necessary conditions for constrained local minima of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda(2x - 5) + 2\mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 2y - 2\lambda y + \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 - 5x - y^2 + 20 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \mu} = 2x + y - 6 = 0$$

Eliminating μ from the first two equations result in

$$2x(\lambda + 1) + 4y(\lambda - 1) = 5\lambda$$

From the fourth condition, we have

$$y = 6 - 2x$$

Substituting for y in the above equation, yields

$$x = \frac{24 - 19\lambda}{10 - 6\lambda}$$

Now substituting for x in the previous equation, we get

$$y = \frac{12 + 2\lambda}{10 - 6\lambda}$$

Substituting for x and y in the third condition (equality constraint) results in an equation in terms of λ

$$\left(\frac{24 - 19\lambda}{10 - 6\lambda}\right)^2 - 5\left(\frac{24 - 19\lambda}{10 - 6\lambda}\right) - \left(\frac{12 + 2\lambda}{10 - 6\lambda}\right)^2 + 20 = 0$$

from which we have the following equation

$$507\lambda^2 - 1690\lambda + 1232 = 0$$

Roots of the above equation are $\lambda = \frac{88}{39}$ and $\lambda = \frac{14}{13}$. Substituting for these values of λ in the expression for x and y, the corresponding extrema are

$$\begin{aligned}
(x, y) &= (\frac{16}{3}, -\frac{14}{3}) \quad &\text{for} \quad \lambda = \frac{88}{39}, \quad \mu = -11.73 \\
(x, y) &= (1, 4) \quad &\text{for} \quad \lambda = \frac{14}{13}, \quad \mu = \frac{8}{13}
\end{aligned}$$

The minimum distance from the cost function is 17, located at point $(1, 4)$, and the maximum distance is 50.22 located at point $(5.333, -4.666)$.

7.8. The fuel-cost functions in \$/h for three thermal plants are given by

$$C_1 = 350 + 7.20P_1 + 0.0040P_1^2$$
$$C_2 = 500 + 7.30P_2 + 0.0025P_2^2$$
$$C_3 = 600 + 6.74P_3 + 0.0030P_3^2$$

where P_1, P_2, and P_3 are in MW. The governors are set such that generators share the load equally. Neglecting line losses and generator limits, find the total cost in \$/h when the total load is

(i) $P_D = 450$ MW
(ii) $P_D = 745$ MW
(iii) $P_D = 1335$ MW

(i) For $P_D = 450$ MW, $P_1 = P_2 = P_3 = \frac{450}{3} = 150$ MW. The total fuel cost is

$$C_t = 350 + 7.20(150) + 0.004(150)^2 + 500 + 7.3(150) + 0.0025(150)^2 + 600 + 6.74(150) + 0.003(150)^2 = 4,849.75 \ \$/\text{h}$$

(ii) For $P_D = 745$ MW, $P_1 = P_2 = P_3 = \frac{745}{3}$ MW. The total fuel cost is

$$C_t = 350 + 7.20\left(\frac{745}{3}\right) + 0.004\left(\frac{745}{3}\right)^2 + 500 + 7.3\left(\frac{745}{3}\right) + 0.0025\left(\frac{745}{3}\right)^2 + 600 + 6.74\left(\frac{745}{3}\right) + 0.003\left(\frac{745}{3}\right)^2 = 7,310.46 \ \$/\text{h}$$

(iii) For $P_D = 1335$ MW, $P_1 = P_2 = P_3 = 445$ MW. The total fuel cost is

$$C_t = 350 + 7.20(445) + 0.004(445)^2 + 500 + 7.3(445) + 0.0025(445)^2 + 600 + 6.74(445) + 0.003(445)^2 = 12,783.04 \ \$/\text{h}$$

7.9 Neglecting line losses and generator limits, determine the optimal scheduling of generation for each loading condition in Problem 7.8
(a) by analytical technique, using (7.33) and (7.31).
(b) using Iterative method. Start with an initial estimate of $\lambda = 7.5$ \$/MWh.
(c) find the savings in \$/h for each case compared to the costs in Problem 7.8 when the generators shared load equally.
Use the **dispatch** program to check your results.

(a) (i) For $P_D = 450$ MW, from (7.33), λ is found to be

$$\lambda = \frac{450 + \frac{7.2}{0.008} + \frac{7.3}{0.005} + \frac{6.74}{0.006}}{\frac{1}{0.008} + \frac{1}{0.005} + \frac{1}{0.006}}$$

$$= \frac{450 + 3483.333}{491.666} = 8.0 \ \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{8.0 - 7.2}{2(0.004)} = 100$$

$$P_2 = \frac{8.0 - 7.3}{2(0.0025)} = 140$$

$$P_3 = \frac{8.0 - 6.74}{2(0.003)} = 210$$

(a) (ii) For $P_D = 745$ MW, from (7.33), λ is found to be

$$\lambda = \frac{745 + 3483.333}{491.666} = 8.6 \;\; \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{8.6 - 7.2}{2(0.004)} = 175$$
$$P_2 = \frac{8.6 - 7.3}{2(0.0025)} = 260$$
$$P_3 = \frac{8.6 - 6.74}{2(0.003)} = 310$$

(a) (iii) For $P_D = 1335$ MW, from (7.33), λ is found to be

$$\lambda = \frac{1335 + 3483.333}{491.666} = 9.8 \;\; \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{9.8 - 7.2}{2(0.004)} = 325$$
$$P_2 = \frac{9.8 - 7.3}{2(0.0025)} = 500$$
$$P_3 = \frac{9.8 - 6.74}{2(0.003)} = 510$$

(b) For the numerical solution using the gradient method, assume the initial value of $\lambda^{(1)} = 7.5$. From coordination equations, given by (7.31), P_1, P_2, and P_3 are

$$P_1^{(1)} = \frac{7.5 - 7.2}{2(0.004)} = 37.5000$$
$$P_2^{(1)} = \frac{7.5 - 7.3}{2(0.0025)} = 40.0000$$
$$P_3^{(1)} = \frac{7.5 - 6.74}{2(0.003)} = 126.6666$$

(i) $P_D = 450$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 450 - (37.5 + 40 + 126.6666) = 245.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{245.8333}{491.6666} = 0.5$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 0.5 = 8.0$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{8.0 - 7.2}{2(0.004)} = 100$$
$$P_2^{(2)} = \frac{8.0 - 7.3}{2(0.0025)} = 140$$
$$P_3^{(2)} = \frac{8.0 - 6.74}{2(0.003)} = 210$$

and

$$\Delta P^{(2)} = 450 - (100 + 140 + 210) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(ii) $P_D = 745$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 745 - (37.5 + 40 + 126.6666) = 540.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{540.8333}{491.6666} = 1.1$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 1.1 = 8.6$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{8.6 - 7.2}{2(0.004)} = 175$$
$$P_2^{(2)} = \frac{8.6 - 7.3}{2(0.0025)} = 260$$
$$P_3^{(2)} = \frac{8.6 - 6.74}{2(0.003)} = 310$$

and

$$\Delta P^{(2)} = 745 - (175 + 260 + 130) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(iii) $P_D = 1335$ MW, the error ΔP from (7.39) is

$$\Delta P^{(1)} = 1335 - (37.5 + 40 + 126.6666) = 1130.8333$$

From (7.37)

$$\Delta\lambda^{(1)} = \frac{1130.8333}{491.6666} = 2.3$$

Therefore, the new value of λ is

$$\lambda^{(2)} = 7.5 + 2.3 = 9.8$$

Continuing the process, for the second iteration, we have

$$P_1^{(2)} = \frac{9.8 - 7.2}{2(0.004)} = 325$$
$$P_2^{(2)} = \frac{9.8 - 7.3}{2(0.0025)} = 500$$
$$P_3^{(2)} = \frac{9.8 - 6.74}{2(0.003)} = 510$$

and

$$\Delta P^{(2)} = 1335 - (325 + 500 + 510) = 0.0$$

Since $\Delta P^{(2)} = 0$, the equality constraint is met in two iterations.

(c)(i) For $P_1 = 100$ MW, $P_2 = 140$ MW, and $P_3 = 210$ MW, the total fuel cost is

$$C_t = 350 + 7.20(100) + 0.004(100)^2 + 500 + 7.3(140) + 0.0025(140)^2 + 600 + 6.74(210) + 0.003(210)^2 = 4,828.70 \ \ \$/\text{h}$$

Compared to Problem 7.8 (i), when the generators shared load equally, the saving is 4,849.75 - 4,828,70 = 21.05 $/h.

(c)(ii) For $P_1 = 175$ MW, $P_2 = 260$ MW, and $P_3 = 310$ MW, the total fuel cost is

$$C_t = 350 + 7.20(175) + 0.004(175)^2 + 500 + 7.3(260) + 0.0025(260)^2 + 600 + 6.74(310) + 0.003(310)^2 = 7,277.20 \ \$/\text{h}$$

Compared to Problem 7.8 (ii), when the generators shared load equally, the saving is 7,310.46 - 7,277.20 = 33.26 \$/h.

(c)(iii) For $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW, the total fuel cost is

$$C_t = 350 + 7.20(325) + 0.004(325)^2 + 500 + 7.3(500) + 0.0025(500)^2 + 600 + 6.74(510) + 0.003(510)^2 = 12,705.20 \ \$/\text{h}$$

Compared to Problem 7.8 (iii), when the generators shared load equally, the saving is 12,783.04 - 12,705.20 = 77.84 \$/h.

To check the results we use the following commands

```
cost = [350  7.2   0.004
        500  7.3   0.0025
        600  6.74  0.003];
disp('(i) Pdt = 450 MW')
Pdt = 450;
dispatch
gencost
disp('(ii) Pdt = 745 MW')
Pdt = 745;
dispatch
gencost
disp('(iii) Pdt = 1335 MW')
Pdt = 1335;
dispatch
gencost
```

The result is

```
(i) Pdt = 450 MW
Incremental cost of delivered power (system lambda)=8.0 $/MWh
Optimal Dispatch of Generation:

  100.0000
  140.0000
```

```
  210.0000
Total generation cost =     4828.70 $/h

(ii) Pdt = 745 MW
Incremental cost of delivered power (system lambda)=8.6 $/MWh
Optimal Dispatch of Generation:

  175.0000
  260.0000
  310.0000
Total generation cost =     7277.20 $/h

(iii) Pdt = 1335 MW
Incremental cost of delivered power (system lambda)=9.80 $/MWh
Optimal Dispatch of Generation:

  325.0000
  500.0000
  510.0000
Total generation cost =    12705.20 $/h
```

7.10. Repeat Problem 7.9 (a) and (b), but this time consider the following generator limits (in MW)

$$122 \leq P_1 \leq 400$$
$$260 \leq P_2 \leq 600$$
$$50 \leq P_3 \leq 445$$

Use the **dispatch** program to check your results.

In Problem 7.9, in part (a) (i), the optimal dispatch are $P_1 = 100$ MW, $P_2 = 140$ MW, and $P_3 = 210$ MW. Since P_1 and P_2 are less that their lower limit, these plants are pegged at their lower limits. That is, $P_1 = 122$, and $P_2 = 260$ MW. Therefore, $P_3 = 450 - (122 + 260) = 68$ MW.

In Problem 7.9, in part (a) (ii), the optimal dispatch are $P_1 = 175$ MW, $P_2 = 260$ MW, and $P_3 = 310$ MW. which are within the plants generation limits.

In Problem 7.9, in part (a) (iii), the optimal dispatch are $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW. Since P_3 exceed its upper limit, this plant is pegged at $P_2 = 445$. Therefore, a load of $1335 - 445 = 890$ MW must be shared between plants 1 and 2, with equal incremental fuel cost give by

$$\lambda = \frac{890 + \frac{7.2}{0.008} + \frac{7.3}{0.005}}{+\frac{1}{0.008} + \frac{1}{0.005}}$$

$$= \frac{890 + 2360}{325} = 10 \ \$/\text{MWh}$$

Substituting for λ in the coordination equation, given by (7.31), the optimal dispatch is

$$P_1 = \frac{10 - 7.2}{2(0.004)} = 350$$
$$P_2 = \frac{10 - 7.3}{2(0.0025)} = 540$$

Since P_1 and P_2 are within their limits the above result is the optimal dispatch.

(b) For part (iii), The iterative method is as follows

In Problem 7.9 (b) part (iii) starting with an initial value of $\lambda^{(1)} = 7.5$, the optimal dispatch was obtained in two iterations as $P_1 = 325$ MW, $P_2 = 500$ MW, and $P_3 = 510$ MW, with $\lambda = 9.8$\$/MWh. Since P_3 exceed its upper limit, this plant is pegged at $P_3 = 445$ and is kept constant at this value. Thus, the new imbalance in power is

$$\Delta P^{(2)} = 1335 - (325 + 500 + 445) = 65$$

From (7.37)

$$\Delta\lambda^{(2)} = \frac{65}{\frac{1}{2(0.004)} + \frac{1}{2(0.0025)}} = \frac{65}{325} = 0.2$$

Therefore, the new value of λ is

$$\lambda^{(3)} = 9.8 + 0.2 = 10$$

For the third iteration, we have

$$P_1^{(3)} = \frac{10 - 7.2}{2(0.004)} = 350$$
$$P_2^{(3)} = \frac{10 - 7.3}{2(0.0025)} = 540$$
$$P_3^{(3)} = 445$$

and

$$\Delta P^{(3)} = 1335 - (350 + 540 + 445) = 0.0$$

$\Delta P^{(3)} = 0$, and the equality constraint is met and P_1 and P_2 are within their limits.

The following commands can be used to obtain the optimal dispatch of generation including generator limits.

To check the results, we use the following commands

```
cost = [350  7.2   0.004
        500  7.3   0.0025
        600  6.74  0.003];

mwlimits =[ 122 400
            260 600
            50 445]
%Pdt = 450;
%Pdt = 745;
Pdt = 1335;
dispatch
gencost
```

The result is

```
Incremental cost of delivered power (system lambda) = 10.0 $/MWh
Optimal Dispatch of Generation:

   350
   540
   445

Total generation cost =   12724.38 $/h
```