Single-Time-Constant (STC) Circuits

This lecture is given as a background that will be needed to determine the frequency response of the amplifiers.

Objectives

- To analyze and understand STC circuits with emphasis on time constant calculations
- To identify different STC configurations: low pass and high pass
- To study the switching performance of the STC circuits

Introduction

Circuits that are composed of, or can be reduced to, one reactive component and one resistance are known as single-time-constant (STC) circuits.

Time Constant (τ) is defined as the time required to charge a reactive element (capacitor or inductor) to 63 percent (actually 63.2 percent) of full charge or to discharge it to 37 percent (actually 36.8 percent) of its initial value.

For an RC circuit, the value of one time constant is expressed mathematically as τ = RC. For RL circuit τ = R/L.

The importance of the STC is appreciated, when we know that the analysis of a complex amplifier circuit can be usually reduced to the analysis of one or more simple STC circuits.

Time Constant Evaluation

The first step in the analysis of an STC circuit is to evaluate its time constant τ.

Example 1:
Find the time constant of the circuit shown in Figure 1.

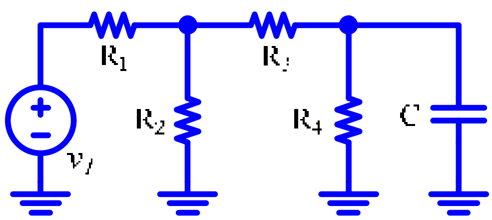


Figure 1 STC circuit for example 1.

Solution:

Apply Thévenin's theorem to find the resistance seen by the capacitor. The solution details are as follow:

Step 1:
Apply Thévenin's theorem to the circuit that contains R_1, R_2, and v_I to obtain R_{th1} and V_{th1} as shown in Figure 2..

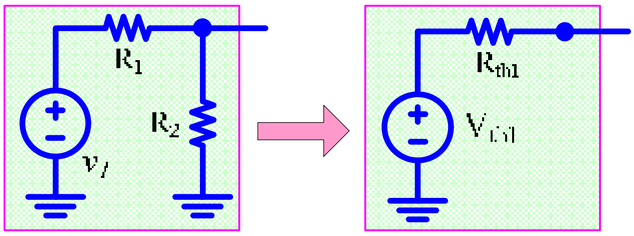


Figure 2 First reduction to the STC in Figure 1 using Thévenin's theorem.

R_{th1} is calculated by reducing v_I to zero. That means both R_1 and R_2 will have the same two nodes.

$\therefore R_{th1} = R_1 \,||\, R_2.$

V_{th1} is the open circuit voltage across R_2 which may be calculated using voltage division.

$\therefore\; V_{th1} = v_i \times R_2 / (R_1 + R_2)$

Step 2:
The STC in Figure 1 may be simplified using Thévenin's circuit in Figure 2 to obtain the simplified circuit in Figure 3.

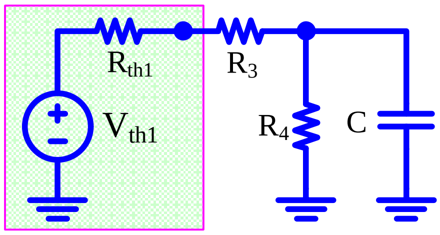


Figure 3 First application of Thévenin's theorem to simplify the STC circuit in Figure 1

Step 3:

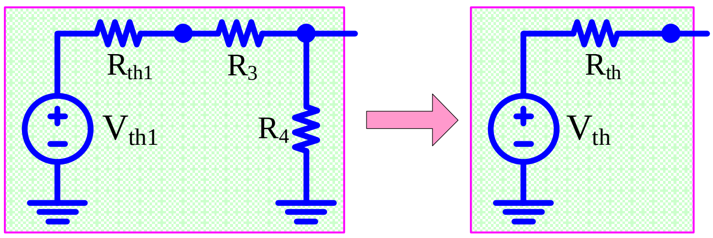


Figure 4 Second reduction to the STC circuit in Figure 1 using Thévenin's theorem.

R_{th} is calculated by reducing v_{th1} to zero. That means both R_{th1} and R_3 will be in series and the resultant resistance will be in parallel with R_4.

$\therefore R_{th} = (R_{th1}+R_3) || R_4 = ((R_1 || R_2)+R_3) || R_4$.

V_{th} is the open circuit voltage across R_4 which may be calculated using voltage division.

$\therefore V_{th}=V_{th1}*R_4/(R_{th1}+R_3+R_4) = v_I*R_2/(R_1+R_2) *R_4/((R_1|| R_2) +R_3+R_4)$

Step 4:

The STC in Figure 2 may be simplified using Thévenin's circuit in Figure 4 to obtain the simplified circuit in Figure 5.

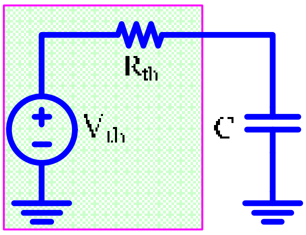


Figure 5 Second application of Thévenin's theorem to simplify the STC circuit in Figure 1

This will result in the resistance seen by the capacitor C as: $[(R_1 || R_2)+R_3] || R_4$.

Thus, $t = \{[(R_1 || R_2)+R_3] || R_4\}C$

Comment: Since the time constant is independent of the sources, first of all set all sources to zero. That means, short-circuit all voltage sources and open circuit all current sources. Then, reduce the circuit as shown in Figure 6.

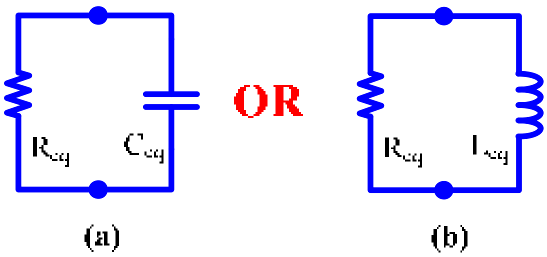


Figure 6 Reduction of STC circuits to a single reactive element and a single resistance

From Figure 6, the time constant will equal to $R_{eq}C_{eq}$ for Figure 5-a, or L_{eq}/R_{eq} for Figure 5-b.

Next example will shed light on this method. (say this in the sound file)

Example 2: Find the time constant of the circuit shown in Figure 7.

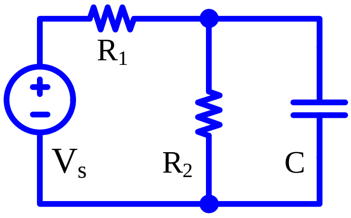


Figure 7 STC Circuit for Example 2

Solution:

Set the source to zero (i.e short circuit v_I) to obtain the circuit in Figure 8-a.
The circuit in Figure 8-a may be further simplified using resistance reduction to obtain the circuit in Figure 8-b.
From Figure 8-b, the resistance seen by the capacitor C will equal to $R_{eq} = R_1 \parallel R_2$
Thus, the time constant will equal to $\tau = (R_1 \parallel R_2)C$

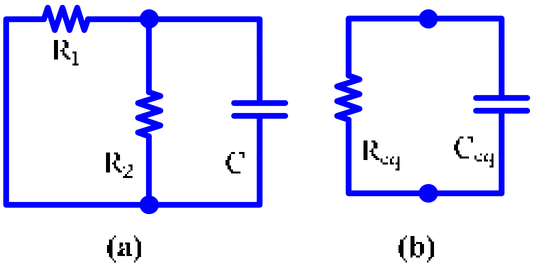


Figure 8 Reduction of the STC circuit in Figure 7 for time constant calculation

Example 3:

Find the time constant of the circuit shown in Fig. 9.

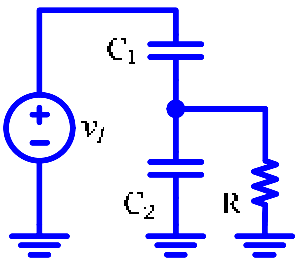


Figure 9 STC circuit for Example 3

Solution:

Set the source to zero (i.e short circuit v_I) to obtain the circuit in Figure 10-a.
The circuit in Figure 10-a may be further simplified using capacitance reduction to obtain the circuit in Figure 10-b.
From Figure 10-b, the capacitance seen by the resistor R will equal to $C_{eq} = C_1 + C_2$
Thus, the time constant will equal to $\tau = \{C_1 + C_2\}R$

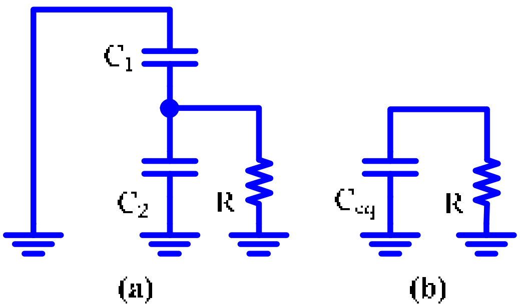


Figure 10 Reduction of the STC circuit in Figure 9 for time constant calculation

Example 4:

Find the time constant of the circuit shown in Figure 11.

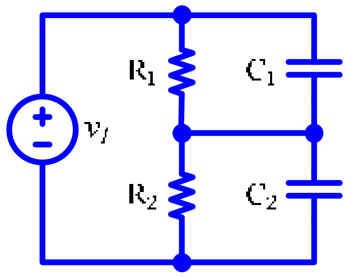


Figure 11 STC circuit for Example 4

Solution:

Set the source to zero (i.e short circuit v_I) to obtain the circuit in Figure 12-a.
The circuit in Figure 12-a may be further simplified by noting that R_1 and R_2 are parallel and C_1 and C_2 are parallel to obtain the circuit in Figure 12-b.
From Figure 12, the equivalent capacitance is $C_{eq} = C_1 + C_2$, and the equivalent resistance is $R_{eq} = R_1 \parallel R_2$ Thus, the time constant will equal to $t = \{C_1 + C_2\}\{R_1 \parallel R_2\}$

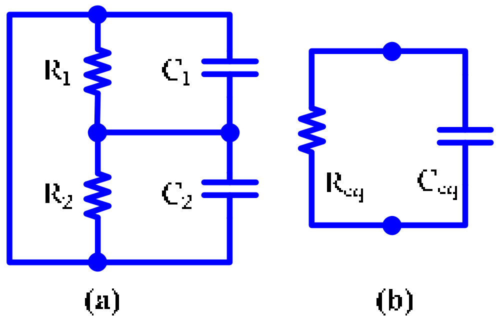


Figure 12 Reduction of the circuit in Figure 11 for time constant calculation

Example 5:

Find the time constant of the circuit shown in Figure 13.

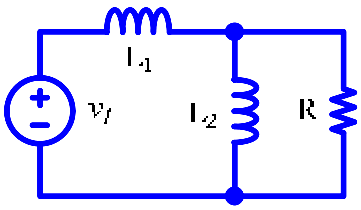


Figure 13 STC circuit for example 5

Solution:
Set the source to zero (i.e short circuit v_I) to obtain the circuit in Figure 14-a.
The circuit in Figure 14-a may be further simplified by noting that L_1 and L_2 are parallel to obtain the circuit in Figure 10-b.
From Figure 10, the inductance seen by the resistor R will equal to $L_{eq} = L_1 \| L_2$ Thus, the time constant will equal to

$$t = \frac{\{L_1 \| L_2\}}{R}$$

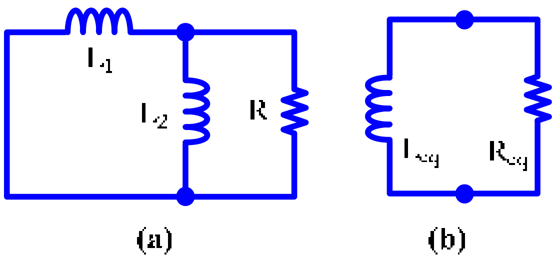


Figure 14 Reduction of the STC circuit in Figure 13 for time constant calculation

Classification of STC Circuits

- STC circuits may be classified into two categories, low-pass (LP) and high-pass (HP) types, with each category displaying distinctly different signal responses.
- Low-Pass circuits pass dc (signal with zero frequency) and attenuate high frequencies, with the transmission being zero at $\omega=\infty$.
- High-Pass circuits pass high frequencies and attenuate low frequencies, with the transmission being zero at dc ($\omega=0$).

- To classify the STC as a LP or HP we may test the output response at either ω=0 or ω=∞.
- At ω=0 capacitors should be replaced by open circuits and inductors should be replaced by short circuits. If the output is zero, the circuit is of the high-pass type otherwise, if the output is finite, the circuit is of the low-pass type.
- At ω=∞ capacitors should be replaced by short circuits and inductors should be replaced by open circuits. If the output is finite, the circuit is of the high-pass type otherwise, if the output is zero, the circuit is of the low-pass type.
- Table 1 provides a summary to the classification test procedure.

Table 1Rules for finding the types of STC circuits

Test at	Replace	Circuit is LP if	Circuit is HP if
ω = 0	C by open circuit & L by short circuit	Output is finite	Output is zero
ω = ∞	C by short circuit & L by open circuit	Output is zero	Output is finite

- Figure 15 shows examples of low-pass STC circuits, and Figure 16 shows examples of high-pass STC circuits.
- Please note that each circuit could be either HP or LP depending on the input and output variable.
- As an exercise please verify the circuits type in Figures 15 and 16using the rules in table 1.
- Detailed frequency response of the STC circuits will be addressed in the next lecture. While the time response to various test signals will be discussed in the next section.

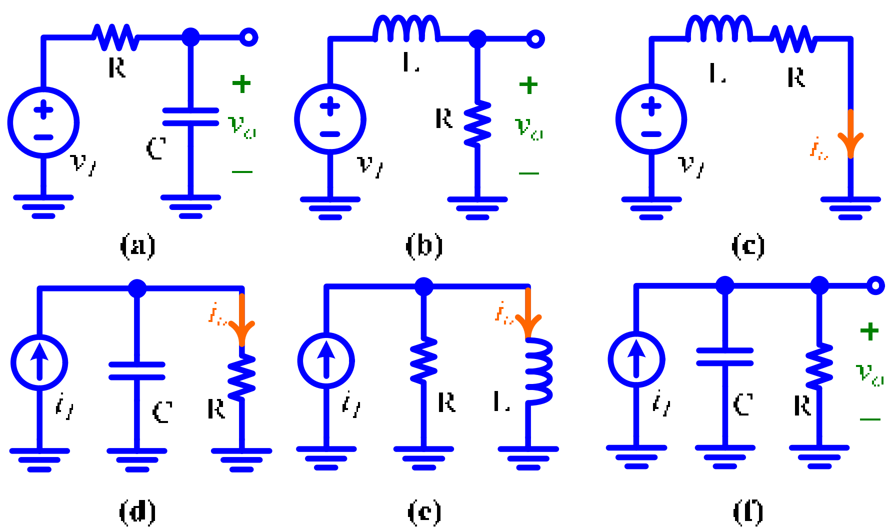


Figure 15 STC circuits of the low-pass type

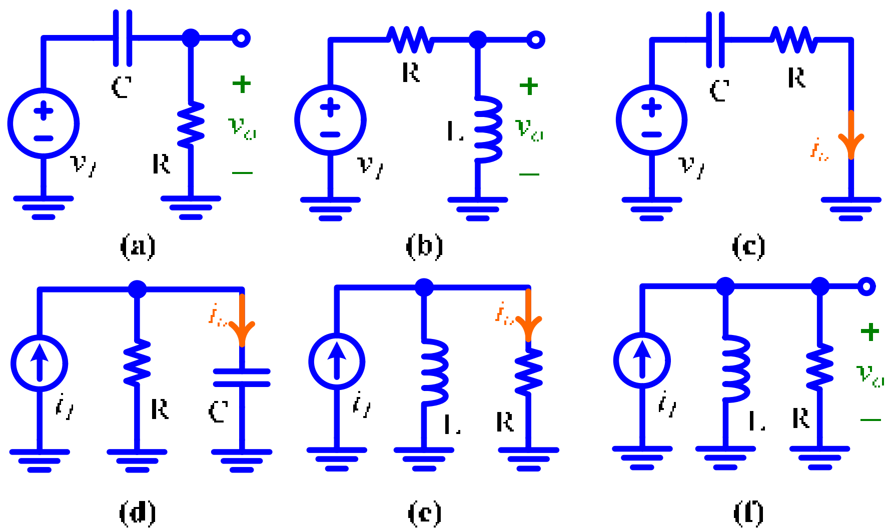


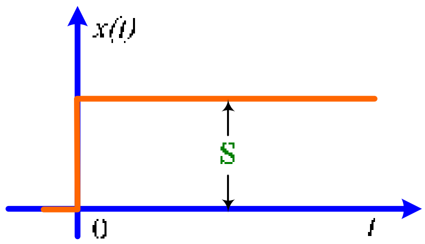
Figure 16 STC circuits of the low-pass type

Time response of STC circuits

Step response of STC Circuits

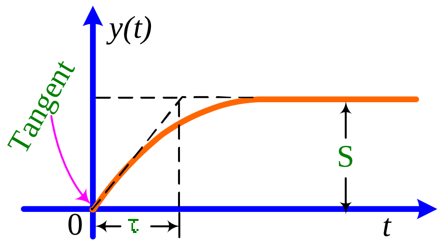
- The first order differential equation describing the system may be solved given the problem initial conditions to yield the required current or voltage.
- Alternatively, we may obtain the required current or voltage without solving any differential equation by finding the following quantities:
  a. the initial value of the capacitor voltage or the inductor current "y_{0+}"
  b. the final value of the capacitor voltage or the inductor current (at $t=\infty$) "y_∞"
  c. the time constant "τ"
  Then use the equation:
  $$y(t) = y_\infty - (y_\infty - y_{0+}).e^{(-t/\tau)}$$
- The step response will depend on the STC type (HP, or LP)
- Figure 17 shows the step response of both HP and LP types.
- For the low-pass circuit
  - $y_{0+} = 0$ (sudden change is considered very high frequency)
  - $y_\infty = S$ (DC signal represents zero frequency)
  - $y(t)=S(1- e^{(-t/\tau)})$ which is shown in Figure 17-b
  - The slope at $t=0$ is S/τ
- For the high-pass circuit
  - $y_{0+} = S$ (sudden change is considered very high frequency)
  - $y_\infty = 0$ (DC signal represents zero frequency)
  - $y(t)=S.e^{(-t/\tau)}$ which is shown in Figure 17-c

- o The slope at t=0 is S/τ



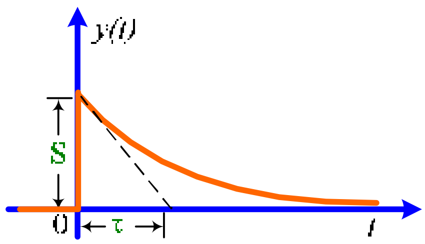
(a)

Figure 17-a A step-function signal of height *S*.



(b)

Figure 17-b The output *y*(*t*) of a low-pass STC circuit excited by a step of height *S*.



(c)

Figure 17-c The output *y*(*t*) of a high-pass STC circuit excited by a step of height *S*.

Pulse response of STC Circuits

- Figure 18 shows a pulse signal with a height "P" and width "T".
- The pulse function may be treated as the sum of two step functions one with a height “P” at time “t=0” and the other with a height “-P” at time “t=T”

- Similar to the step response, the pulse response will depend on the STC type (LP or HP)

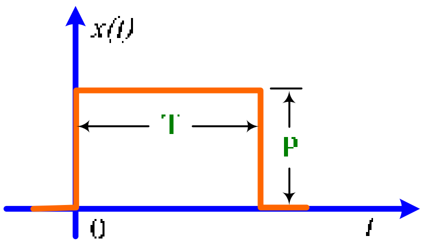


Figure 18 A pulse signal with a height "P" and width "T"

a) Low Pass circuit response

- The response will depend on the ratio between the pulse width "T" and the time constant "τ".
- Figure 19 shows the response for three different time constant ratio cases
- Since low pass circuit passes the DC faithfully, the area under the curve will be constant equal to T times P as the input signal.
- If τ<<T, the output will be similar to the input with smoothed edges (remember that sudden changes represent very high frequencies)
- As shown in Figure 19-a, the rise time is defined as the time required for the output to rise from 10% to 90% of the pulse height. Similarly, the fall time is defined as the time required for the output to fall from 90% to 10% of the pulse height.
- From the exponential equation we can easily prove that $t_r = t_f = 2.2 * \tau$
- An interesting case is when τ >>T where the circuit will act as an integrator.

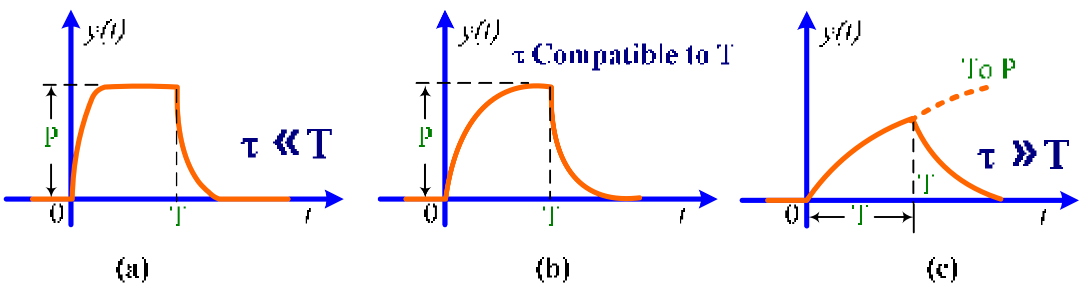


Figure 19 The output *y*(*t*) of a low-pass STC circuit excited by a pulse in Figure 18

b) High Pass circuit response

- Similar to the low pass circuit. The response will depend on the ratio between the pulse width "T" and the time constant "τ".
- Figure 20 shows the response for three different time constant ratio cases
- Since high pass circuit has an infinite attenuation to DC, the area under the curve will equal to zero.
- If τ>>T, the output will be similar to the input with small tilt and zero average (positive area will equal to the negative area)
- The tilt ΔP may be calculated by noting the slope of the step response at t=0 is P/τ. That means ΔP may be approximated by the slope times T = PT/τ.
- An interesting case is when τ <<T where the circuit will act as a differentiator.

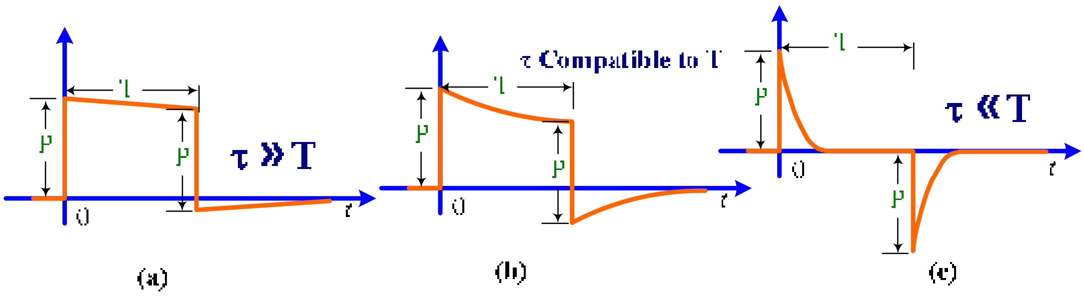


Figure 20 The output *y*(*t*) of a high-pass STC circuit excited by a pulse in Figure 18

s-Domain Analysis, and Bode Plots

This lecture is given as a background that will be needed to determine the frequency response of the amplifiers.

Objectives

- To study the frequency response of the STC circuits
- To appreciate the advantages of the logarithmic scale over the linear scale
- To construct the Bode Plot for the different STC circuits
- To draw the Bode Plot of the amplifier gain given its transfer characteristics

Introduction

In the last lecture we examined the time response of the STC circuits to various test signals. In that case the analysis is said to be carried in the time-domain.
The analysis and design of any electronic circuit in general or STC circuits in particular may be simplified by considering other domains rather than the time domain. One of the most common domains for electronic circuits' analysis is the s-domain. In this domain the independent variable is taken as the complex frequency "s" instead of the time.
As we said the importance of studying the STC circuits is that the analysis of a complex amplifier circuit can be usually reduced to the analysis of one or more simple STC circuits.

s-Domain Analysis

The analysis in the s-domain to determine the voltage transfer function may be summarized as follows:

- Replace a capacitance C by an admittance sC, or equivalently an impedance $1/sC$.
- Replace an inductance L by an impedance sL.
- Use usual circuit analysis techniques to derive the voltage transfer function $T(s) \equiv V_o(s)/ V_i(s)$

Example 1: Find the voltage transfer function $T(s) \equiv Vo(s)/Vi(s)$ for the STC network shown in Figure 1?

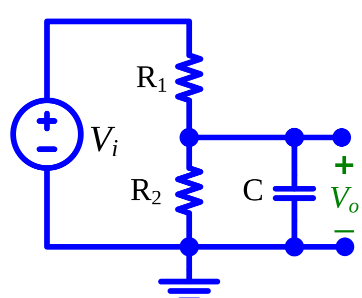


Figure 1 STC circuit to be analyzed using s-Domain

Solution:

Step 1:
Replace the capacitor by impedance equal to 1/SC as shown in Figure 2. Note that both V_i and V_o will be functions of the complex angular frequency (s)

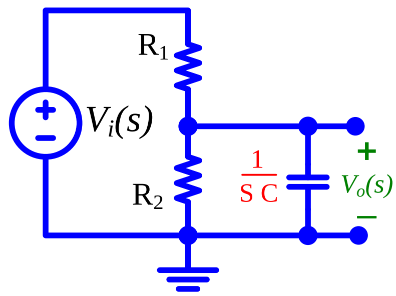


Figure 2 The STC in figure 1 with the capacitor replaced by an impedance 1/SC

Step 2:
Use nodal analysis at the output node to find $V_o(s)$.

$$\frac{V_o(s)}{1/sC} + \frac{V_o(s)}{R_2} + \frac{V_o(s) - V_i(s)}{R_1} = 0$$

$$\Rightarrow V_o(s)\left(sC + \frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_i(s)}{R_1}$$

$$\Rightarrow T(s) = \frac{V_o(s)}{V_i(s)} = \frac{1/R_1}{\left(sC + \frac{1}{R_1} + \frac{1}{R_2}\right)}$$

$$\therefore T(s) = \frac{1/CR_1}{s + 1/C(R_1 \| R_2)}$$

As an exercise try to use the impedance reduction, and the voltage divider rule, or any other method to calculate T(s).

- In most cases $T(s)$ will reveal many useful facts about the circuit performance.
- For physical frequencies s may be replaced by jw in $T(s)$. The resulting transfer function $T(jw)$ is in general a complex quantity with its:
  - Magnitude gives the magnitude (or transmission) response of the circuit
  - Angle gives the phase response of the circuit

Example 2: For Example 1 assuming sinusoidal driving signals; calculate the magnitude and phase response of the STC circuit in Figure 1?
Solution:

Step 1:

Replace *s* by *jw* in *T*(*s*) to obtain *T*(*jω*)

$$\because T(s) = \frac{1/CR_1}{s + 1/C(R_1 \,||\, R_2)}$$

$$\therefore T(j\omega) = \frac{1/CR_1}{jw + 1/C(R_1 \,||\, R_2)}$$

Step 2: The magnitude and angle of *T*(*jω*) will give the magnitude response and the phase response respectively as shown below:

$$|T(j\omega)| = \frac{1/CR_1}{\sqrt{\omega^2 + \left(1/C(R_1 \,||\, R_2)\right)^2}}$$

$$\theta(j\omega) = \angle\left(T(j\omega)\right) = 0 - \arctan[\omega C(R_1 \,||\, R_2)]$$

Poles and Zeros

In General for all the circuits dealt with in this course, T(s) can be expressed in the form

$$T(s) = \frac{N(s)}{D(s)}$$

where both N(s) and D(s) are polynomials with real coefficients and an order of m and n respectively

- The **order of the network** is equal to n
- For real systems, the degree of *N(s)* (or m) is always less than or equal to that of *D(s)*(or n). Think about what happens when $s \rightarrow \infty$.

An alternate form for expressing T(s) is

$$T(s) = a_m \frac{(s - Z_1)(s - Z_2)...(s - Z_m)}{(s - P_1)(s - P_2)...(s - P_m)}$$

where a_m is a multiplicative constant; Z_1, Z_2, …, Z_m are the roots of the numerator polynomial (N(s)); P_1, P_2, …, P_n are the roots of the denominator polynomial (D(s)).

Poles — roots of $D(s) = 0$ { $P_1, P_2, \ldots, P_n$} are the points on the s-plane where |T| goes to ∞.

Zeros — roots of $N(s) = 0$ { $Z_1, Z_2, \ldots, Z_m$} are the points on the s-plane where |T| goes to 0.

- The poles and zeros can be either real or complex. However, since the polynomial coefficients are real numbers, the complex poles (or zeros) must occur in conjugate pairs.
- A zero that is pure imaginary ($\pm j\omega_z$) cause the transfer function $T(j\omega)$ to be exactly zero (or have transmission null) at $\omega=\omega_z$.
- Real zeros will not result in transmission nulls.
- For stable systems all the poles should have negative real parts.
- For s much greater than all the zeros and poles, the transfer function may be approximated as $T(s) \cong a_m/s^{n-m}$. Thus the transfer function have (n-m) zeros at $s=\infty$.

Example 3: Find the poles and zeros for the following transfer function T(s)? What is the order of the network represented by T(s)? What is the value of T(s) as s approaches infinity?

$$T(s) = \frac{s(s^2+100)}{(s^2+4s+13)(s+10)}$$

Solution:

Poles : $-2 \pm j3$ and -10 which are the points on s-plane where |T| goes to ∞.

Zeros : 0 and $\pm j10$ which are the points on s-plane where |T| goes to 0.

The network represented by T(s) is a third order which is the order of the denominator

$$\lim_{s\to\infty} |T(s)| = 1$$

Plotting Frequency Response

Problem with scaling

As seen before, the frequency response equations (magnitude and phase) are usually nonlinear— some square within a square root, etc. and some arctan function!

The most difficult problem with linear scale is the limited range as illustrated in the following figure

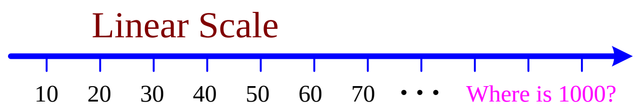


Figure 3 Linear scale range limitation

If the x-axis is plotted in log scale, then the range can be widened.

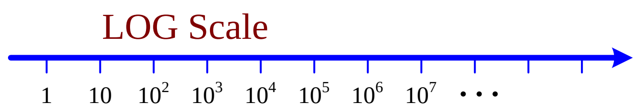


Figure 4 Log scale representation

As we can see from Figure 4, the log scale may be used to represent small quantities together with large quantities. A feature not visible with linear range.

Bode technique: asymptotic approximation

The other problem now is the non-linearity of the magnitude and phase equations. A simple technique for obtaining an approximate plots of the magnitude and the phase of the transfer function is known as **Bode plots** developed by H. Bode.
The Bode technique is particularly useful when all the poles and zeros are real.

To understand this technique let us draw the magnitude and the phase Bode plots of a STC circuit transfer function given by $T(s)=1/(1+s/\omega_p)$; where $\omega_p = 1/CR$.
Please note that this transfer function represent a low-pass STC circuit. Also, T(s) represents a simple pole.

Simple Pole Magnitude Bode Plot Construction

Replace s in T(s) by jω to obtain T(jω)

$$\Rightarrow T(j\omega) = \frac{1}{1 + j\omega/\omega_p}$$

Find the magnitude of T(jω)

$$\Rightarrow |T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_p)^2}}$$

Take the log of both sides and multiply by 20

$$\Rightarrow 20\log\left(\left|T(j\omega)\right|\right) = -10\log\left(1+(\omega/\omega_p)^2\right)$$

Define y=20log|T(ω)| and x=log(ω)
The unit of y is the decibel (dB)

In terms of x and y we have

$$y = -10\log\left(1+(\omega/\omega_p)^2\right)$$

Now for large and small values of ω, we can make some approximation

- $\omega >> \omega_p$: $y \cong -10 \log (\omega/\omega_p)^2 = -20 \log (\omega/\omega_p) = -20x + 20 \log(\omega_p)$
  which represent a straight line of slope = -20. The unit of the slope will be dB/decade (unit of y axis per unit of x axis)
- $\omega << \omega_p$: $y \cong -10 \log (1) = 0$
  Which represents a horizontal line.

Finally, if we plot y versus x, then we get straight lines as asymptotes for large and small ω . Note that the approximation will be poor near ω_p with a maximum error of 3 dB (10 log(2) at $\omega=\omega_p$)

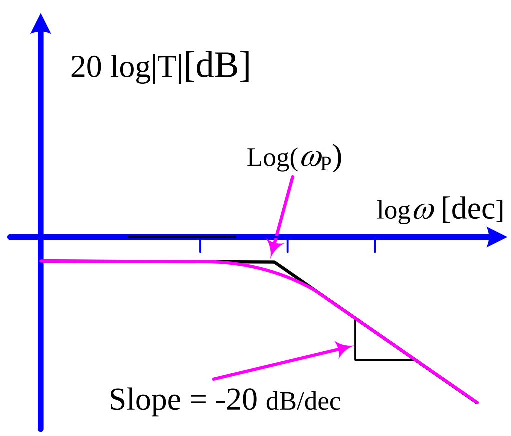


Figure 5 Bode Plot for the magnitude of a simple pole

Simple Pole Phase Bode Plot Construction

Replace s in T(s) by jω to obtain T(jω)

$$\therefore T(jw) = \frac{1}{1+\frac{j\omega}{\omega_p}}$$

Find the angle of T(jω)

$\Rightarrow \theta(\omega) = \angle\left(T(j\omega)\right) = 0 - \arctan\left(\omega / \omega_p\right)$

Now for large and small values of ω, we can make some approximation

- $\omega >> \omega_p$: $\theta(\omega) \cong -\arctan(\infty) = -\pi/2 = -90°$
  we can assume that much greater (>>) is equal to 10 times
- $\omega << \omega_p$: $\theta(\omega) \cong -\arctan(0) = -0°$
  we can assume that much less (<<) is equal to 0.1 times
- For the frequencies between 0.1 ω_p and 10 ω_p we may approximate the phase response by straight line which will have a slope of -45°/decade with a value equal to -45° at $\omega=\omega_p$

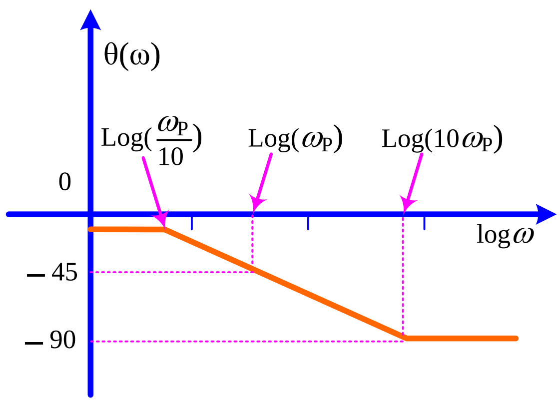


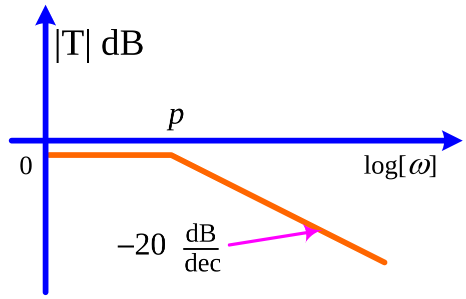
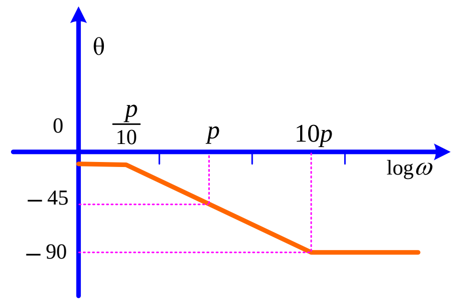
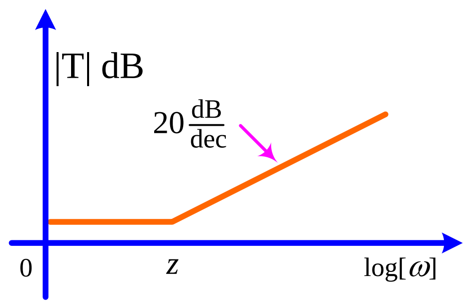
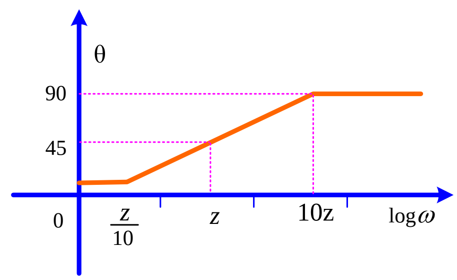
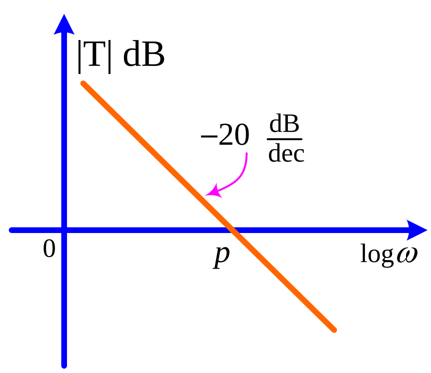
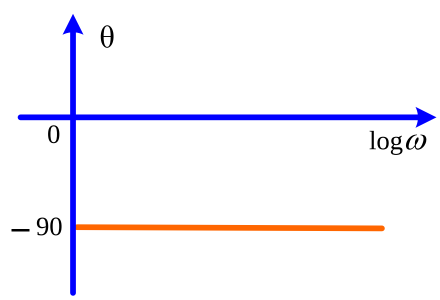
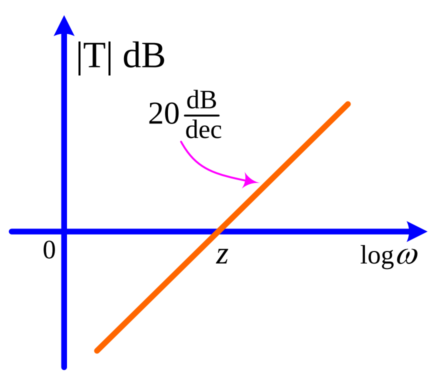
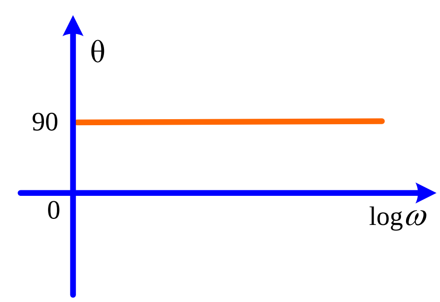
Figure 6 Bode Plot for the phase of a simple pole

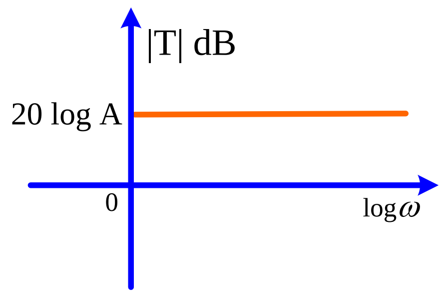
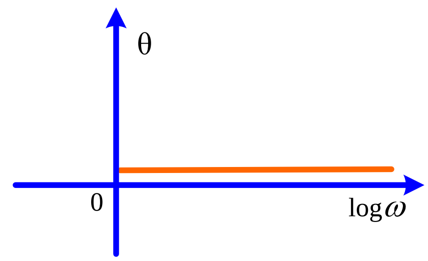
Bode Plots: General Technique

Since Bode plots are log-scale plots, we may plot any transfer function by adding together simpler transfer functions which make up the whole transfer function.

The standard forms of Bode plots may be summarized as shown in Table 1.

Table 1 Bode plots standard forms

<table>
<tr><th>Form</th><th>Equation</th><th>Magnitude Bode plot</th><th>Phase Bode plot</th></tr>
<tr><td>Simple pole</td><td>$\dfrac{1}{1+{}^{s}\!/\!{}_{p}}$</td><td></td><td></td></tr>
<tr><td>Simple zero</td><td>$1+{}^{s}\!/\!{}_{z}$</td><td></td><td></td></tr>
<tr><td>Integrating pole</td><td>$\dfrac{1}{{}^{s}\!/\!{}_{p}}$</td><td></td><td></td></tr>
<tr><td>Differentiating zero</td><td>${}^{s}\!/\!{}_{z}$</td><td></td><td></td></tr>
</table>

Form	Equation	Magnitude Bode plot	Phase Bode plot
Constant	A		

Example 4: A circuit has the following transfer function

$$H(s)=\frac{1000\left(1+\frac{s}{200\pi}\right)}{\frac{s}{2\pi}\left(1+\frac{s}{60000\pi}\right)}$$

Sketch the magnitude and phase Bode plots.

Solution:

By referring to Table 1 we can divide H(s) into four simpler transfer functions as shown below:

$$H(s)=\frac{\overbrace{1000}^{\text{constant}}\overbrace{\left(1+\frac{s}{200\pi}\right)}^{\text{simple zero}}}{\underbrace{\frac{s}{2\pi}}_{\text{differentiating pole}}\underbrace{\left(1+\frac{s}{60000\pi}\right)}_{\text{simple pole}}}$$

The total Bode plot may be obtained by adding the four terms as shown in Figure 7. Please note that f=ω/2π.

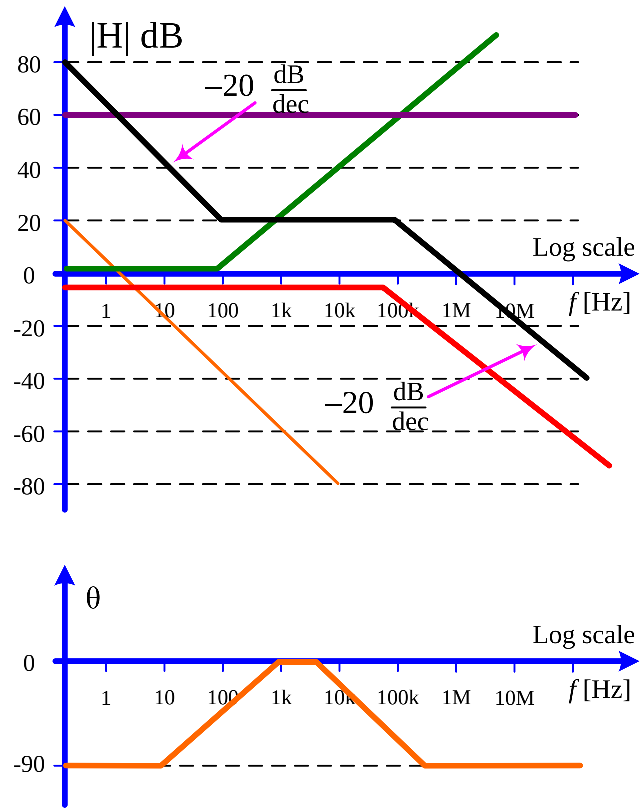


Figure 7 Magnitude and Phase Bode Plots of Example 4