Problem 6.9

Because of spherical symmetry, $\vec{E} = E_r \vec{a}_r$ and thus $V = V(r)$.

Therefore, La Place's equation in spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta}) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Is simplified to $\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{dV}{dr}) = 0$

Solving for $V \quad \Rightarrow \quad \frac{d}{dr}(r^2 \frac{dV}{dr}) = 0 \quad \Rightarrow \quad (r^2 \frac{dV}{dr}) = c_1 \quad \Rightarrow \quad \frac{dV}{dr} = c_1 r^{-2}$

$$V(r) = -c_1 r^{-1} + c_2 \quad \text{for} \quad 0.1 \le r \le 2$$

Applying the boundary conditions, namely $V(0.1) = 0$ and $V(2) = 100$, we can evaluate the arbitrary constants c_1 and c_2.

$$V(0.1) = -10c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 10c_1$$

$$\therefore V(r) = -c_1 r^{-1} + 10c_1 = c_1(10 - r^{-1})$$

$$V(2) = c_1(10 - 2^{-1}) = 100 \quad \Rightarrow \quad c_1 = \frac{100}{9.5} = 10.526$$

$$\therefore V(r) = 10.526(10 - r^{-1}) \quad \text{for} \quad 0.1 \le r \le 2$$

$$\vec{E} = -\nabla V \quad \Rightarrow \quad \vec{E} = -[\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi] = -\frac{dV}{dr}\vec{a}_r$$

$$\therefore \vec{E} = -\frac{10.526}{r^2}\vec{a}_r \quad \text{for} \quad 0.1 \le r \le 2$$

(notice that the electric field is directed from the higher potential towards the lower potential)

$$\vec{D} = \varepsilon_0 \vec{E} = -\frac{10.526\varepsilon_0}{r^2}\vec{a}_r = -\frac{9.316\times 10^{-11}}{r^2}\vec{a}_r \quad \text{for} \quad 0.1 \le r \le 2$$

Problem 6.11

Because of cylindrical symmetry, $\vec{E} = E_\rho \vec{a}_\rho$ and thus $V = V(\rho)$.

Therefore, La Place's equation in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial V}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Is simplified to $\frac{1}{\rho}\frac{d}{d\rho}(\rho \frac{dV}{d\rho}) = 0$

Solving for V $\Rightarrow$ $\frac{d}{d\rho}(\rho \frac{dV}{d\rho}) = 0$ $\Rightarrow$ $\rho \frac{dV}{d\rho} = c_1$ $\Rightarrow$ $\frac{dV}{d\rho} = c_1 \rho^{-1}$

$V(\rho) = c_1 \ln \rho + c_2$ for $0.02 \le \rho \le 0.06$

Applying the boundary conditions, namely $V(0.02) = 60$ and $V(0.06) = -20$, we can evaluate the arbitrary constants c_1 and c_2.

$V(0.02) = c_1 \ln 0.02 + c_2 = 60$ $\Rightarrow$ $-3.9120 c_1 + c_2 = 60$ (1)

$V(0.06) = c_1 \ln 0.06 + c_2 = -20$ $\Rightarrow$ $-2.8134 c_1 + c_2 = -20$ (2)

Solving (1) and (2) $\Rightarrow$ $c_1 = -72.82$ & $c_2 = -224.88$

$V(\rho) = -72.82 \ln \rho - 224.88$ for $0.02 \le \rho \le 0.06$

$$\vec{E} = -\nabla V \quad \Rightarrow \quad \vec{E} = -[\nabla V = \frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z] = -\frac{dV}{d\rho}\vec{a}_\rho$$

$\therefore \vec{E} = \frac{72.82}{\rho}\vec{a}_\rho$ for $0.02 \le \rho \le 0.06$

(clearly $\vec{E}$ points from the high to the low potential)

$V(0.04) = -72.82 \ln 0.04 - 224.88 = 9.454 V$

$$\vec{E}\Big|_{\rho=0.04} = \frac{72.82}{0.04}\vec{a}_\rho = 1820.5 \vec{a}_\rho \; V/m$$

$$\vec{D}\Big|_{\rho=0.04} = 1820.5 \varepsilon_0 \vec{a}_\rho = 1.611 \times 10^{-8} \vec{a}_\rho \; C/m^2$$

Problem 6.18

$$E = \frac{V_o}{d}$$

(In the region between the plates $\vec{E}$ is uniform and has the same value in both air and the dielectric)

Since $w_e = \frac{1}{2}\varepsilon E^2$

The electrostatic energy density in air is $w_{e1} = \frac{1}{2}\varepsilon_0(\frac{V_o}{d})^2$ (uniform energy density)

The electrostatic energy density in the dielectric air is $w_{e2} = \frac{1}{2}\varepsilon_0\varepsilon_r(\frac{V_o}{d})^2$ (uniform energy density)

The stored electrostatic is given by $W_e = \int_V w_e dv = w_{e1}V_1 + w_{e2}V_2$

(Integration becomes multiplication because we are integrating uniform quantities).

$$W_e = \int_V w_e dv = w_{e1}V_1 + w_{e2}V_2 \quad \Rightarrow \quad W_e = \frac{1}{2}\varepsilon_0(\frac{V_o}{d})^2(L-x)ad + \frac{1}{2}\varepsilon_0\varepsilon_r(\frac{V_o}{d})^2 xad$$

$$W_e = \frac{1}{2}\varepsilon_0(\frac{V_o}{d})^2 ad[L + \varepsilon_r x - x]$$

The magnitude of the restoring force F is given by:

$$F = \left|\frac{dW_e}{dx}\right| = \frac{1}{2}\varepsilon_0(\frac{V_o}{d})^2 ad(\varepsilon_r - 1)$$

$$F = \frac{\varepsilon_0(\varepsilon_r - 1)aV_o^2}{2d}$$

Problem 6.20

$$C = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

$$C = \frac{2\pi \times (3.5\varepsilon_0) \times 1000}{\ln\frac{2\times 10^{-3}}{1\times 10^{-3}}} = \frac{7000\pi\varepsilon_0}{\ln 2} = 2.81\times 10^{-7}\ F$$

Which is the capacitance of 1km length of cable.

Thus, the capacitance per km of the cable equals $2.81\times 10^{-7}\ F/km$

Problem 6.22

a) $C = \dfrac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$ $\Rightarrow$ $C = \dfrac{4\pi(2.25\varepsilon_0)}{\dfrac{1}{0.05} - \dfrac{1}{0.1}} = \dfrac{9\pi\varepsilon_0}{10} = 2.5023 \times 10^{-11}\ F$

b) $Q = CV_o$ $\Rightarrow$ $Q = (2.5023 \times 10^{-11}) \times 80 = 2.002 \times 10^{-9}\ C$

The surface charge density is uniform $\Rightarrow$ $\rho_s = \dfrac{Q}{S} = \dfrac{Q}{4\pi a^2} = \dfrac{2.002 \times 10^{-9}}{4\pi(0.05)^2} = 6.37 \times 10^{-8}\ C/m^2$