Problem 5.17

a) $\oint_S \vec{D}.\vec{ds} = Q \quad \Rightarrow \quad D4\pi r^2 = Q \quad \Rightarrow \quad \varepsilon E 4\pi r^2 = Q \quad \Rightarrow \quad E = \frac{Q}{4\pi\varepsilon r^2}$

$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\vec{a}_r$ (because of spherical symmetry, $\vec{E}$ has only an r component)

For $r < a \quad \Rightarrow \quad Q = \int_V \rho_o dv \quad \xrightarrow{\text{Uniform Charge Density}} \quad Q = \rho_o V = \rho_o \frac{4\pi}{3} r^3$

$\therefore \quad \vec{E} = \frac{\rho_o r}{3\varepsilon_0 \varepsilon_r}\vec{a}_r \quad for\ r < a$

For $r > a \quad \Rightarrow \quad Q = \rho_o V = \rho_o \frac{4\pi}{3} a^3$

$\therefore \quad \vec{E} = \frac{\rho_o a^3}{3\varepsilon_0 r^2}\vec{a}_r \quad for\ r > a$

$$V_P = -\int_{ref}^{P} \vec{E}.\vec{dl}$$

$$V(0,0,0) = -\int_{r=\infty}^{0} E_r dr = -\int_{r=\infty}^{a} E_r dr - \int_{r=a}^{0} E_r dr$$

(the integral is split because the expression for $\vec{E}$ is different in the two regions $r < a$ and $r > a$).

$$V(0,0,0) = -\int_{r=\infty}^{a} \frac{\rho_o a^3}{3\varepsilon_0 r^2} dr - \int_{r=a}^{0} \frac{\rho_o r}{3\varepsilon_0 \varepsilon_r} dr \quad \Rightarrow \quad V(0,0,0) = \left.\frac{\rho_o a^3}{3\varepsilon_0 r}\right|_{\infty}^{a} - \left.\frac{\rho_o r^2}{6\varepsilon_0 \varepsilon_r}\right|_{a}^{0}$$

$$V(0,0,0) = \frac{\rho_o a^2}{3\varepsilon_0} + \frac{\rho_o a^2}{6\varepsilon_0 \varepsilon_r} \quad \Rightarrow \quad V(0,0,0) = \frac{2\rho_o a^2 \varepsilon_r + \rho_o a^2}{6\varepsilon_0 \varepsilon_r} = \frac{\rho_o a^2 (2\varepsilon_r + 1)}{6\varepsilon_0 \varepsilon_r}$$

b) The potential at the surface of the sphere is clearly $-\int_{r=\infty}^{a} \frac{\rho_o a^3}{3\varepsilon_0 r^2} dr = \frac{\rho_o a^2}{3\varepsilon_0}$

Problem 5.23

a) $\vec{J} = 10^4(x^2+y^2)\vec{a}_z \quad \Rightarrow \quad \vec{J}(-3,4,6) = 10^4(9+16)\vec{a}_z = 25\times10^4\vec{a}_z \ A/m^2$

b) Using the current continuity equation $\nabla\vec{J} = -\dfrac{\partial\rho_v}{\partial t}$

$$\frac{\partial\rho_v}{\partial t} = -\nabla\vec{J} = -\nabla[10^4(x^2+y^2)\vec{a}_z] = 0$$

c) $I = \int_S \vec{J}.\vec{ds} \quad \Rightarrow \quad I = \int_S 10^4(x^2+y^2)\vec{a}_z.(dxdy\vec{a}_z)$

[The surface is circular, so it is easier to use cylindrical coordinates than rectangular coordinates].

Using $x^2+y^2=\rho^2$ and $\vec{ds} = \rho d\rho d\phi\vec{a}_z$

$$I = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{5\times10^{-3}}(10^4\rho^2)(\rho d\rho d\phi) = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{5\times10^{-3}}10^4\rho^3 d\rho d\phi = 10^4(2\pi)\left.\frac{\rho^4}{4}\right|_0^{5\times10^{-3}} = 10^4(2\pi)\frac{(5\times10^{-3})^4}{4}$$

$\therefore \quad I = 9.82\times10^{-6} = 9.82\mu A$

Problem 5.34

b) The boundary is the $x-z$ plane. Therefore the tangential and normal components of $\vec{E}_1$ are:

$\vec{E}_{1t} = 10\vec{a}_x + 12\vec{a}_z$ and $\vec{E}_{1n} = -6\vec{a}_y$

$\vec{E}_{2t} = \vec{E}_{1t} = 10\vec{a}_x + 12\vec{a}_z$

$D_{2n} - D_{1n} = \rho_s = 0$ (the boundary is charge free)

$$\therefore \quad D_{2n} = D_{1n} \quad \Rightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \quad \Rightarrow \quad E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} \quad \Rightarrow \quad E_{2n} = \frac{3\varepsilon_0}{4.5\varepsilon_0} \times (-6) = -4$$

$\therefore \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = 10\vec{a}_x - 4\vec{a}_y + 12\vec{a}_z$ (Answer)

To find the angle that $\vec{E}_2$ makes with respect to the y - axis:

$\vec{E}_{2t} = 10\vec{a}_x + 12\vec{a}_z \quad \Rightarrow \quad E_{2t} = \sqrt{10^2 + 12^2} = \sqrt{244} = 15.62$ (Magnitude of the <u>tangential</u> component)

$\vec{E}_{2n} = -4\vec{a}_y \quad \Rightarrow \quad E_{2n} = 4$ (Magnitude of the <u>normal</u> component)

$\therefore \ \theta_2 = \tan^{-1}(\frac{E_{2t}}{E_{2n}}) = \tan^{-1}(\frac{15.62}{4}) = 75.64^o$ (Answer)

c) $w_e = \frac{1}{2}\varepsilon E^2$

In region 1 $\Rightarrow$ $w_{e1} = \frac{1}{2}(3\varepsilon_0)[10^2 + (-6)^2 + 12^2] = 420\varepsilon_0 = 3.72\ nJ/m^3$

In region 2 $\Rightarrow$ $w_{e2} = \frac{1}{2}(4.5\varepsilon_0)[10^2 + (-4)^2 + 12^2] = 585\varepsilon_0 = 5.177\ nJ/m^3$

Problem 6.1

a) $V = 5x^3y^2z \quad \Rightarrow \quad \vec{E} = -\nabla V = -(15x^2y^2z\vec{a}_x + 10x^3yz\vec{a}_y + 5x^3y^2\vec{a}_z)$

$\vec{E}(-3,1,2) = -[15\times 9\times 1\times 2\vec{a}_x + 10\times(-27)\times 1\times 2\vec{a}_y + 5\times(-27)\times 1\vec{a}_z]$

$\vec{E}(-3,1,2) = -270\vec{a}_x - 540\vec{a}_y - 135\vec{a}_z$

b) $\vec{D} = \varepsilon\vec{E} = -2.25\varepsilon_0(15x^2y^2z\vec{a}_x + 10x^3yz\vec{a}_y + 5x^3y^2\vec{a}_z)$

$= -33.75\varepsilon_0 x^2y^2z\vec{a}_x - 22.5\varepsilon_0 x^3yz\vec{a}_y - 11.25\varepsilon_0 x^3y^2\vec{a}_z$

Using Gauss's law in differential form $\rho_v = \nabla\vec{D}$

$\rho_v = -67.5\varepsilon_0 xy^2z - 22.5\varepsilon_0 x^3z$

$\rho_v(-3,1,2) = 67.5\varepsilon_0 \times 3\times 2 + 22.5\varepsilon_0 \times 27\times 2$

$\rho_v(-3,1,2) = 1620\varepsilon_0 = 14.34\times 10^{-9} = 14.34\,\text{nC/m}^3$

Another way to solve this problem is to use Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad \Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

$$\Rightarrow \quad \frac{\partial^2(5x^3y^2z)}{\partial x^2} + \frac{\partial^2(5x^3y^2z)}{\partial y^2} + \frac{\partial^2(5x^3y^2z)}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

$$\Rightarrow \quad 30xy^2z + 10x^3z + 0 = -\frac{\rho_v}{\varepsilon} \quad \Rightarrow \quad \rho_v = -\varepsilon(30xy^2z + 10x^3z)$$

$\rho_v(-3,1,2) = -2.25\varepsilon_0[(30(-3)(2) - 10(27)(2)]$

$\rho_v(-3,1,2) = 2.25\varepsilon_0(180 + 540) = 14.34\ \text{nC/m}^3$

Which is the same answer as above.

Problem 6.5

La Place's equation states that $\nabla^2 V = 0$

a) $V = e^{-5x}\cos(13y)\sinh 12z$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Note that $\frac{d(\sinh x)}{dx} = \cosh x$ and $\frac{d(\cosh x)}{dx} = \sinh x$

$$\nabla^2 V = 25e^{-5x}\cos(13y)\sinh 12z - 169e^{-5x}\cos(13y)\sinh 12z + 144e^{-5x}\cos(13y)\sinh 12z$$

$$\therefore \nabla^2 V = 0$$

b) $V = \frac{z\cos\phi}{\rho}$ & $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}(-\rho\frac{z\cos\phi}{\rho^2}) - \frac{z}{\rho^3}\cos\phi + 0 \quad \Rightarrow \quad \nabla^2 V = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\frac{z\cos\phi}{\rho}) - \frac{z}{\rho^3}\cos\phi$$

$$\nabla^2 V = \frac{z\cos\phi}{\rho^3} - \frac{z}{\rho^3}\cos\phi = 0$$

c) $V = \frac{30\cos\theta}{r^2}$ & $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\phi^2}$

$$\nabla^2 V = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{60\cos\theta}{r^3}) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{30\sin\theta}{r^2}) + \frac{1}{r^2\sin\theta}(0)$$

$$\nabla^2 V = -\frac{60\cos\theta}{r^2}\frac{\partial}{\partial r}(\frac{1}{r}) - \frac{30}{r^4\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$\nabla^2 V = \frac{60\cos\theta}{r^4} - \frac{30}{r^4\sin\theta}(2\sin\theta\cos\theta)$$

$$\nabla^2 V = \frac{60\cos\theta}{r^4} - \frac{60\cos\theta}{r^4} = 0$$