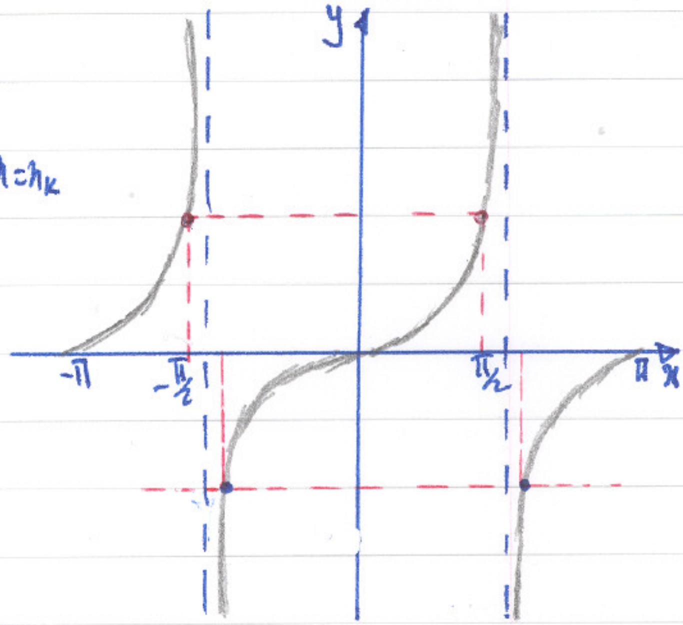
Solution To Problem 3.4-2

Homework # 3

9- Problem 3.4-2:

This problem can be solved directly using the following:

$$f_Y(y) = \sum_k \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=x_k}$$



$$f_X(x) = \begin{cases} \frac{1}{\pi} & , -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & , \text{elsewhere} \end{cases}$$

It can be seen from the graph whether $y \geq 0$ or $y < 0$, there are two roots. In both cases, there is only one which belongs to the interval $[-\pi/2, \pi/2]$ for which the pdf of the random variable is defined. Therefore, the pdf of the random variable for $y \in \mathbb{R}$ is given by:

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=\tan^{-1}\left(\frac{y}{a}\right)}$$

Since $y = a \tan x$, then $x = \tan^{-1}\left(\frac{y}{a}\right)$, $a > 0$

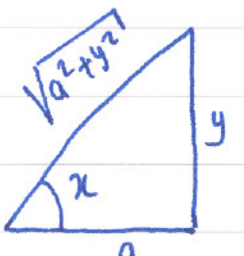
Hence, $\dfrac{dy}{dx} = \dfrac{a}{\cos^2 x}$

$$\therefore \quad f_Y(y) = \left. \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \right|_{x = \tan^{-1}\left(\frac{y}{a}\right)}$$

$f_X\left(\tan^{-1}\left(\frac{y}{a}\right)\right) = \dfrac{1}{\pi}$ since $x = \tan^{-1}\left(\frac{y}{a}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\cos^2\left(\tan^{-1}\left(\frac{y}{a}\right)\right) = \frac{a^2}{\left\{\sqrt{a^2+y^2}\right\}^2} = \frac{a^2}{a^2+y^2}$$

since



$\tan x = \dfrac{y}{a}$

$$\therefore \quad \cos^2 x = \left\{\frac{a}{\sqrt{a^2+y^2}}\right\}^2 = \frac{a^2}{a^2+y^2}$$

$$\therefore \quad f_Y(y) = \frac{1/\pi}{\left\{\dfrac{a}{\cos^2\left[\tan^{-1}\left(\frac{y}{a}\right)\right]}\right\}}$$

$$f_Y(y) = \frac{a}{\pi(a^2+y^2)}, \quad y \in \mathbb{R}$$