Electrical Engineering Department
Final (062)
EE 662
Date: Monday June 4, 2007
Time: 7:00 PM -10:00 PM

Q1) *(10 Marks)* Let $\hat{x}$ be the optimum minimum mean-square estimate (MMSE) of x given a random variable y. Consider the random variable $z = f(x)$. Can we claim that the MMSE estimate of z is $\hat{z} = f(\hat{x})$? Justify your answer by either proving it or providing a counter example.

Q2) *(30 Marks)* The Kroneckor product is important for the transient analysis of adaptive filters. Here we try to explore some of its properties. In the following, all matrices are square of size M.

a) Starting from the formal definition of the Kroneckor product, prove that

$$(A \otimes B)^* = A^* \otimes B^*$$

b) Now let Q_1 and Q_2 be unitary matricies. Is $Q_1 \otimes Q_2$ unitary? Justify your answer.

b) Let R and Z be positive definite. Define $H = R \otimes Z$

- **(i)** Find the eigenvalue decomposition of H in terms of the eigenvalue decompositions of R and Z.
- **(ii)** Prove that H is positive definite if R and Z are. Is the converse true?

Q2) *(35 Marks)* Consider the LMS algorithm described by

$$w_i = w_{i-1} + \mu u_i^* e(i) \quad w_{-1} = 0 \tag{1}$$

$$e(i) = d(i) - u_i w_{i-1} \tag{2}$$

In the lecture, we studied the mean-square behavior of the weight error vector. Here we would like to study the behavior of the mean of the weight error vector.

a) Assume that $d(i)$ is given by

$$d(i) = u_i w^o + v(i)$$

Prove that the weight error vector $\tilde{w}_i = w^o - w_i$ satisfies the recursion

$$\tilde{w}_i = (I - \mu u_i^* u_i)\, \tilde{w}_{i-1} - \mu u_i^* v(i)$$

a) Assume the sequence $\{u_j\}$ is iid and is independent of the zero-mean iid sequence $\{v(j)\}$.

- **(i)** Prove that u_i and $\tilde{w_{i-1}}$ are independent.
- **(ii)** Derive a recursion for the behavior of the weight error vector.
- **(iii)** Find a necessary and sufficient condition for the stability of $E[\tilde{w}_i]$.

(iv) What is the steady-state value of $E[\tilde{w}_i]$ assuming the recursion is stable?

(v) What is the steady-state value of $E[w_i]$?

b For this part, assume that u_i and $\tilde{w}_i$ are independent but that u_i and v_i are not. Rather, for all i

$$E[u_i^* v(i)] = b \neq 0$$

(i) Derive a recursion for the behavior of the weight error vector.

(ii) Find a necessary and sufficient condition for the stability of $E[\tilde{w}_i]$.

(iii) What is the steady-state value of $E[\tilde{w}_i]$ assuming the recursion is stable?

Q4) *(15 Marks)*

a) Consider the least squares problem

$$\min_w \|y - Hw\|^2$$

and assume that the matrix H^*H is invertible. Show that $H^*\mathcal{P}_H^\perp = 0$. Use this to show that $\mathcal{P}_H^\perp$ is orthogonal to any vector in the column span of H.

b) Consider the least squares problem

$$\min_w a_1\|y_1 - H_1 w\|^2_{\Pi_1} + a_2\|y_2 - H_2 w\|^2_{\Pi_2} + |b^* w|^2$$

where Π_1 and Π_2 are positive definite and the scalars a_1 and a_2 are postive. Use the weighted norm properties to write this problem in the form

$$\min_w \|y - Hw\|$$

c) Consider the least squares problem.

$$\min_{w_1} \min_{w_2} \|y_1 - H_1 w_1\|^2 + \|y_2 - H_2 w_2\|^2$$

Find the solution of this problem assuming that both H_1 and H_2 are full rank.

Q5) *(10 Marks)* In adaptive filtering anaylsis, we need to evaluate various weighted moments. Let u_i be a zero mean random vector with autocorrelation matrix R.

a) Find the value of $E[\|u_i\|^2]$ in terms of R.

b) Find the value of $E[\|u_i\|^2_{R^M}]$ in terms of R.