

Formula Sheet

Algebra of Sets

- 1) $A \cap B = B \cap A, A \cup B = B \cup A$
- 2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- 3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- 4) $\overline{(B \cap C)} = \overline{B} \cup \overline{C}$

Joint and Conditional Probability

- 5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- 6) $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- 7) For mutually exclusive events B_n with $\bigcup_{n=1}^N B_n = S$,

$$P(A) = \sum_{n=1}^N P(A|B_n)P(B_n)$$
- 8) Bayes' Rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
- 9) For Independent events A and B
 $P(A \cap B) = P(A)P(B)$
 $P(A|B) = P(A), P(B|A) = P(B)$

Permutations and Combinations

- 10) No. of Permutations of r elements from n : $P_r^n = \frac{n!}{(n-r)!}$
- 11) No. Combinations of r elements from n : $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Bernoulli Trials

- 12) For an event A with $P(A) = p$

$$P\{\text{exactly } k \text{ occurrences in } N \text{ trials}\} = \binom{N}{k} p^k (1-p)^{N-k}$$

Distribution Function (CDF)

- 13) $F_X(x) = P\{X \leq x\}$

Properties of CDFs

- 14) $F_X(-\infty) = 0$ 15) $F_X(\infty) = 1$
- 16) $0 \leq F_X(x) \leq 1$ 17) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$
- 18) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
- 19) $F_X(x^+) = F_X(x)$

Properties of PDFs

- 20) $0 \leq f_X(x)$ for all x 21) $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$$22) F_X(x) = \int_{-\infty}^x f_X(\alpha) d\alpha$$

$$23) P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(\alpha) d\alpha$$

CDF of a Discrete RV

- 24) For discrete X with values x_i with $P\{x_i\} = P\{X = x_i\}$

$$f_X(x) = \sum_{i=1}^N P\{x_i\} \delta(x - x_i)$$

$$F_X(x) = \sum_{i=1}^N P\{x_i\} u(x - x_i)$$

A Gaussian RV has

$$25) f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$$

$$26) F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^x e^{-\frac{(\alpha-a_x)^2}{2\sigma_x^2}} d\alpha$$

A Binomial RV has

$$27) f_X(x) = \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} \delta(x - k)$$

$$28) F_X(x) = \sum_{k=0}^N \binom{N}{k} p^k (1-p)^{N-k} u(x - k)$$

A Uniform RV has

$$29) f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$$30) F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$$

A Poisson RV with

- 31) $b = \lambda \cdot T$, λ = rate of arrivals, T = period of interest

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x - k)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x - k)$$

An Exponential RV has

$$32) f_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b} e^{-\frac{x-a}{b}} & x > a \end{cases}$$

$$33) F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{x-a}{b}} & x > a \end{cases}$$

An Rayleigh RV has

$$34) f_X(x) = \begin{cases} 0 & x < a \\ \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & x \geq a \end{cases}$$

$$35) F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{(x-a)^2}{b}} & x \geq a \end{cases}$$

Conditional Distribution

$$36) F_X(x|B) = P\{X \leq x | B\} = \frac{P\{X \leq x \cap B\}}{P(B)}$$

$$37) f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

Expected value of an RV or a function of an RV

$$38) E[X] = \bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$39) E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

Moments about the origin and Central Moments

$$40) m_n = E[X^n] \quad 41) \mu_n = E\left[(X - \bar{X})^n\right]$$

$$42) \text{Variance } \sigma_X^2 = \mu_2$$

$$43) \text{Skew } \mu_3 = E\left[(X - \bar{X})^3\right]$$

Characteristic Function

$$44) \Phi_X(\omega) = E\left[e^{j\omega X}\right] \quad 45)$$

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

Transformation of RVs

46) If $Y = T(X)$ and T is monotonic function

$$f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|$$

47) If $Y = T(X)$ and T is non-monotonic function

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \right|_{x=x_n}}$$

48) If $Y = T(X)$ is a constant y_1 over some period $x_1 \leq X \leq x_2$, the RV Y has discrete component with

$$P\{Y = y_1\} = F_X(x_1) - F_X(x_2) = \int_{x_1}^{x_2} f_X(x) dx$$

Joint Distributions

$$49) F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\}$$

$$50) f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Properties of Joint CDFs

$$51) F_{X,Y}(-\infty, -\infty) = 0, \quad F_{X,Y}(\infty, \infty) = 1$$

$$52) F_{X,Y}(-\infty, y) = 0, \quad F_{X,Y}(x, -\infty) = 0$$

$$53) 0 \leq F_{X,Y}(x,y) \leq 1$$

54) $F_{X,Y}(x,y)$ is non-decreasing of x and y

$$55) P\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\} = F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2)$$

$$56) F_{X,Y}(x, \infty) = F_X(x), \quad F_{X,Y}(\infty, y) = F_Y(y)$$

Properties of Joint PDFs

$$57) f_{X,Y}(x,y) \geq 0$$

$$58) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$$

$$59) F_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy dx$$

$$F_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy$$

$$60) P\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{X,Y}(x,y) dy dx$$

$$61) f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy,$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

Conditional Density and Distribution

$$62) f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

63) For discrete RV X and Y such that X takes values x_i

$$f_X(x|Y = y_k) = \sum_{i=1}^N \frac{P(x_i, y_k)}{P(y_k)} \delta(x - x_i)$$

$$F_X(x|Y = y_k) = \sum_{i=1}^N \frac{P(x_i, y_k)}{P(y_k)} u(x - x_i)$$

64) For continuous RVs X and Y

$$f_X(x|Y = y) = f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Statistical Independence

65) X and Y are statistically independent if any is true

$$F_{X,Y}(x,y) = F_X(x)F_Y(y),$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Distribution of Sum of RVs

66) for $W = X + Y \rightarrow F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_{X,Y}(x,y) dy dx$

67) If X and Y are Statistically Independent

$$f_w(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x) dx$$

Expectations of Multiple RVs

68) $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy dx$

69) $m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y) dy dx$

70) $R_{XY} = m_{11} = E[XY]$

71) If X and Y are Statistically Independent
 $R_{XY} = E[XY] = E[X] \cdot E[Y]$

72) X and Y orthogonal if $R_{XY} = E[XY] = 0$

73) $\mu_{nk} = E[(X - \bar{X})^n (Y - \bar{Y})^k]$

74) $C_{XY} = \mu_{11} = E[(X - \bar{X})(Y - \bar{Y})] = R_{XY} - \bar{X}\bar{Y}$

75) $\rho = \frac{C_{XY}}{\sigma_X \sigma_Y}, \quad -1 \leq \rho \leq 1$

Characteristic Function of Joint RVs

76) $\Phi_{X,Y}(\omega_1, \omega_2) = E[e^{j\omega_1 X + j\omega_2 Y}]$

77) $m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k} \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1^n \partial \omega_2^k} \right|_{\omega_1=0, \omega_2=0}$

Jointly Gaussian Random Variables

78)
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2} \right]}$$

Estimation of Mean, Power, and Variance

79) $\widehat{X}_N = \frac{1}{N} \sum_{n=1}^N X_n, \quad \widehat{X}_N^2 = \frac{1}{N} \sum_{n=1}^N X_n^2$

$$\widehat{\sigma}_N^2 = \frac{1}{N-1} \sum_{n=1}^N (X_n - \widehat{X}_N)^2$$

Density and Distribution Function of Random of Processes

80) $F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$

$$f_X(x_1; t_1) = dF_X(x_1; t_1) / dx_1$$

81) $F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$

$$f_X(x_1, x_2; t_1, t_2) = \partial^2 F_X(x_1, x_2; t_1, t_2) / (\partial x_1 \partial x_2)$$

Stationarity of Random Processes

82) 1st-order stat: $f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$

83) 2nd-order stat:

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

84) If $X(t)$ is 1st and 2nd-order stat. $\rightarrow X(t)$ is strictly stat.

85) $R_{XX}(t_2, t_2) = E[X(t_1)X(t_2)]$

86) $C_{XX}(t_1, t_2) = R_{XX}(t_2, t_2) - E[X(t_1)]E[X(t_2)]$

87) $X(t)$ is wide-sense stationary if

$$E[X(t)] = \bar{X} = \text{constant}$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

Time Averages and Ergodicity

88) Time average is given by $A[\cdot] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [\cdot] dt$

89) $\bar{x} = A[x(t)]$

90) $\mathfrak{R}_{xx}(\tau) = A[x(t)x(t + \tau)]$

91) If $\bar{x} = \bar{X}$ and $\mathfrak{R}_{xx}(\tau) = R_{XX}(\tau) \rightarrow X(t)$ is Ergodic

Properties of $R_{XX}(\tau)$

92) $R_{XX}(0) = E[X^2(t)], \quad 93) R_{XX}(\tau) = R_{XX}(-\tau)$

94) $|R_{XX}(\tau)| \leq R_{XX}(0)$

Transmission of a random process through a linear filter

If a w.s.s $X(t)$ is fed to a filter $h(t)$ to produce $Y(t)$

95) $Y(t) = h(t) * X(t)$

96) $\mu_Y = \mu_X H(0), \quad R_{YY}(\tau) = h(\tau) * h(-\tau) * R_{XX}(\tau)$

97) $S_{XX}(f) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j2\pi f \tau} d\tau,$

$$R_{XX}(\tau) = \int_{-\infty}^{+\infty} S_{XX}(f) e^{j2\pi f \tau} df$$

98) $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$

Properties of $S_{XX}(f)$

99) $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau = \text{DC power}$

100) $R_{XX}(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(f) df = \text{Total power}$

101) $S_{XX}(f) \geq 0$ for all f

102) $S_{XX}(f) = S_{XX}(-f) \quad 103) S_{XX}(f)$ is real