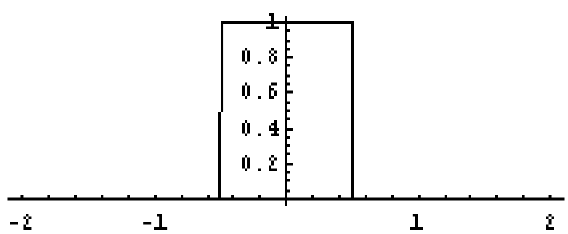
A first look at the Fourier transform

We're about to make the transition from Fourier series to the Fourier transform. 'Transition' is the appropriate word, for in the approach we'll take the Fourier transform emerges as we pass from periodic to nonperiodic functions To make the trip we'll view a nonperiodic function (which can be just about anything) as a limiting case of a periodic function as the period becomes longer and longer. It's actually not so immediate. It takes a little extra tinkering to coax the Fourier transform out of the Fourier series, but it's an interesting approach.[1]

Let's take a specific, simple, and important example. Consider the 'rect' function ('rect' for 'rectangle') defined by

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| \geq 1/2 \end{cases}$$

Here's the graph, which is not very complicated.

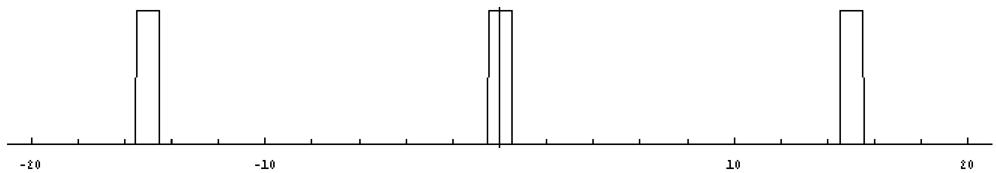


$\Pi(t)$ is even, centered at the origin, and has width 1. Later we'll consider shifted and scaled versions. You can think of $\Pi(t)$ as modeling a switch that is on for one second and off for the rest of the time. Π is also called, variously, the 'top hat' function, the 'indicator' function for the interval $(-1/2, 1/2)$, or the 'characteristic' function for the interval $(-1/2, 1/2)$.

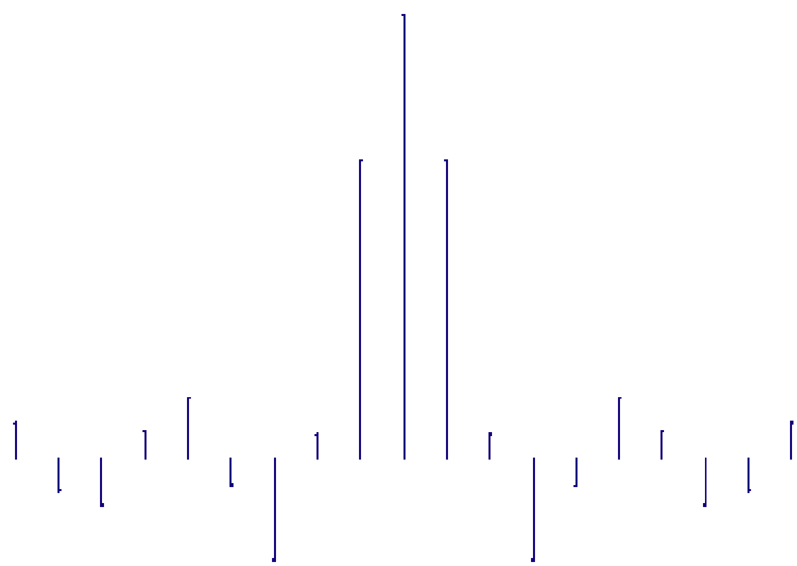
While we have defined $\Pi(\pm 1/2) = 0$ other common conventions are either to have $\Pi(\pm 1/2) = 1$ or $\Pi(\pm 1/2) = 1/2$. And some people don't define Π at $\pm 1/2$ at all, leaving two holes in the domain. I don't want to get dragged into this dispute. It almost never matters, though for some purposes the choice $\Pi(\pm 1/2) = 1/2$ makes the most sense. We'll deal with this on an exceptional basis if and when it comes up.

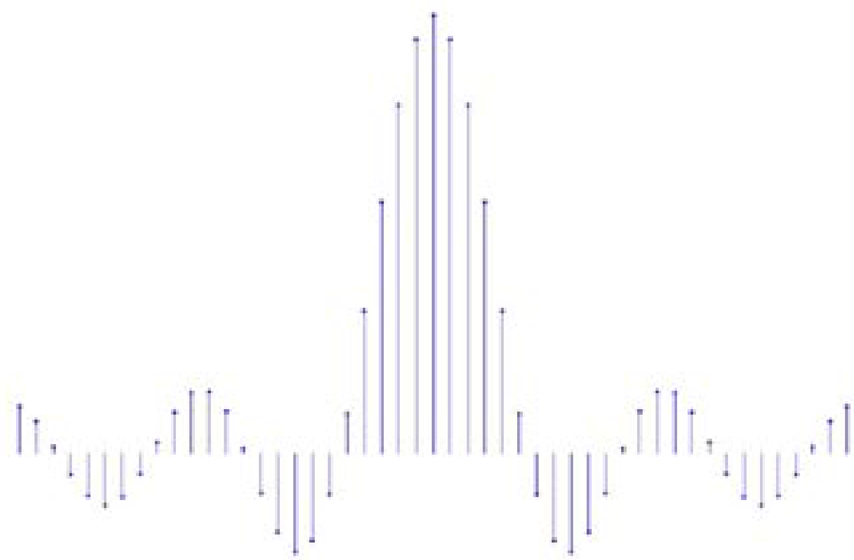
[1] As an aside, I don't know if this is the best way of motivating the definition of the Fourier transform, but I don't know a better way and most sources you're likely to check will just present the formula as a done deal. It's true that, in the end, it's the formula and what we can do with it that we want to get to, so if you don't find the (brief) discussion to follow to your tastes, I am not offended.

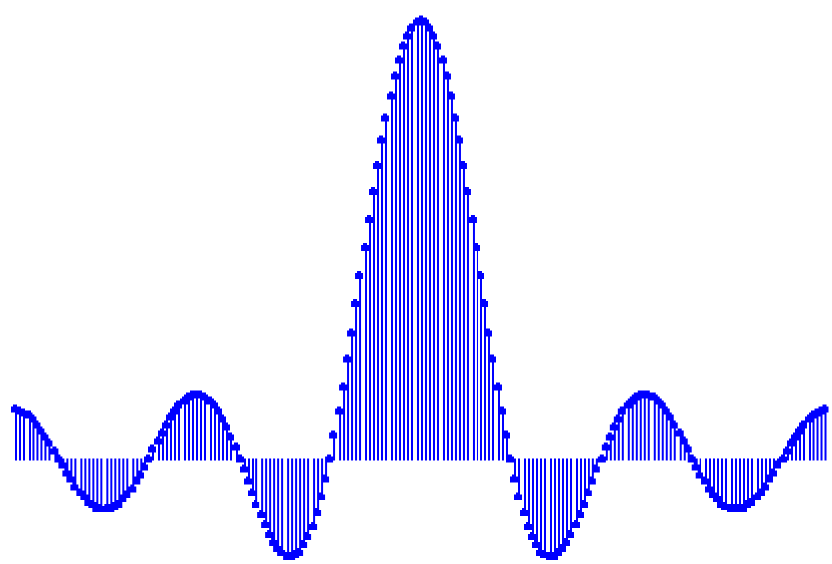
$\Pi(t)$ is not periodic. It doesn't have a Fourier series. In the first problem set you experimented a little with periodizations of the triangle function, and I want to do that again with Π but for a specific purpose. As a periodic version of $\Pi(t)$ we repeat the nonzero part of the function at regular intervals, separated by (long) intervals where the function is zero. We can think of such a function when we flip a switch on for a second but do so repeatedly, and we keep it off for a long time in between the times it's on. Here's a plot of $\Pi(t)$ periodized to have period 15.



Here are some plots of the Fourier coefficients of periodized rectangle functions when the period is 15, 30 and 100, respectively. Because the function is real and even in each case the Fourier coefficients are real, so these are plots of the actual coefficients, not their square magnitudes.







We see that as the period increases the frequencies are getting closer and closer together and it looks as though the coefficients are tracking some definite curve. (But there's an important issue here of vertical scaling – we'll see.) We can analyze what's going on in this particular example, and combine that with some general statements to lead us on.

Recall that for a general function $f(t)$ of period T the Fourier series has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T},$$

so that the frequencies are $0, \pm 1/T, \pm 2/T, \ldots$. Points in the spectrum are spaced $1/T$ apart and, indeed, in the pictures above the spectrum is getting more tightly packed as the period T increases. The n'th Fourier coefficient is given by

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt.$$

We can calculate this for $\Pi(t)$:

$$\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T}\,\Pi(t)\,dt \\
&= \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T}\cdot 1\,dt \\
&= \frac{1}{T}\left[\frac{1}{-2\pi i n/T} e^{-2\pi i n t/T}\right]_{t=-1/2}^{t=1/2} \\
&= \frac{1}{2\pi i n}\left(e^{\pi i n/T} - e^{-\pi i n/T}\right) \\
&= \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right).
\end{aligned}$$

Now, although the spectrum is *indexed* by n (it's a discrete set of points), the points in the spectrum are n/T, $n = 0, \pm 1, \pm 2, \ldots$, and it's more helpful to think of the 'spectral information' (the value of c_n) as a transform of Π *evaluated* at the points n/T. Write this, provisionally, as

$$(\text{Transform of periodized }\Pi)\left(\frac{n}{T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right).$$

We're almost there, but not quite. If you're dying to just take a limit as $T \to \infty$ consider that, for each n, if T is very large then n/T is very small and

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) \quad \text{is about size} \quad \frac{1}{T} \quad (\text{remember } \sin\theta \approx \theta \text{ if } \theta \text{ is small}).$$

In other words, for each n this so-called transform,

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) \quad \text{tends to 0 like } 1/T.$$

To compensate for this we scale up by T, that is, consider instead

$$(\text{Scaled transform of periodized }\Pi)\left(\frac{n}{T}\right) = T\,\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) = \frac{\sin\left(\frac{\pi n}{T}\right)}{\frac{\pi n}{T}}.$$

In fact, the plots of the *scaled* transforms are what I showed you, above.

Next, if T is large then we can think of replacing the closely packed discrete points n/T by a continuous variable, say s, so that with $s = n/T$ we would then write, approximately,

$$(\text{Scaled transform of periodized } \Pi)(s) = \frac{\sin \pi s}{\pi s}.$$

What does this procedure look like in terms of the integral formula? Simply

$$(\text{Scaled transform of periodized } \Pi)(\frac{n}{T}) = T \cdot c_n = T \cdot \frac{1}{T} \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt = \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\, dt.$$

If we now think of $T \to \infty$ as having the effect of replacing the discrete variable n/T by the continuous variable s, as well as pushing the limits of integration to $\pm\infty$, then we may write for the (limiting) transform of Π the integral expression

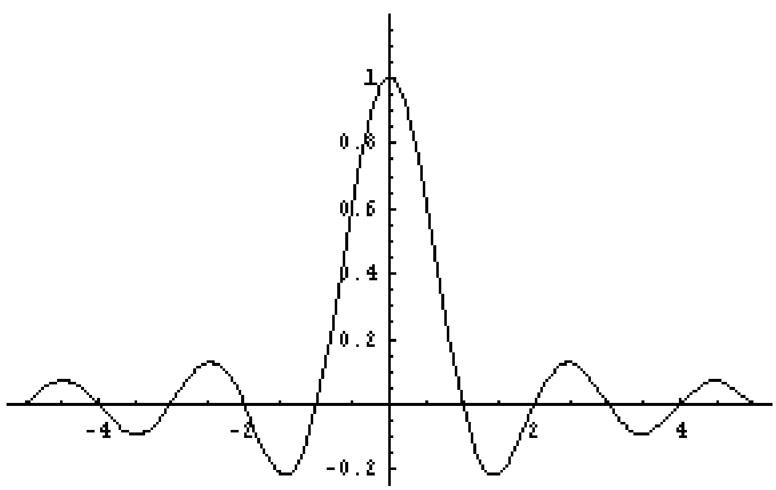
$$\widehat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} \Pi(t)\, dt.$$

Behold, the Fourier transform is born!

Let's calculate the integral. (We know what the answer is, because we saw the discrete form of it earlier.)

$$\begin{aligned} \widehat{\Pi}(s) &= \int_{-\infty}^{\infty} e^{-2\pi i s t} \Pi(t)\, dt \\ &= \int_{-1/2}^{1/2} e^{-2\pi i s t} \cdot 1\, dt \\ &= \frac{\sin \pi s}{\pi s}. \end{aligned}$$

Here's a graph. You can now certainly see the continuous curve that the plots of the discrete, scaled Fourier coefficients are shadowing.



The function $\sin \pi x/\pi x$ (written now with a generic variable x) comes up so often in this subject that it's given a special symbol:

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x},$$

read ‘sink’. Note that

$$\operatorname{sinc} 0 = 1$$

by virtue of the famous limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

It’s fair to say that many EE’s see the sinc function in their dreams.

How general is this? We would be led to the same idea – scale the Fourier coefficients by T – if we had started off periodizing just about any old function with the intention of letting $T \to \infty$. Suppose $f(t)$ is zero outside of $|t| \le 1/2$. (Any interval will do, we just want to suppose a function is zero outside some interval so we can periodize.) We periodize $f(t)$ to have period T and compute the Fourier coefficients:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi i n t/T} f(t)\,dt.$$

How big is this? We can estimate

$$\begin{aligned}
|c_n| &= \frac{1}{T}\left|\int_{-1/2}^{1/2} e^{-2\pi i n t/T} f(t)\,dt\right| \\
&\le \frac{1}{T}\int_{-1/2}^{1/2} |e^{-2\pi i n t/T}|\,|f(t)|\,dt \\
&= \frac{1}{T}\int_{-1/2}^{1/2} |f(t)|\,dt \\
&= \frac{A}{T}
\end{aligned}$$

where

$$A = \int_{-1/2}^{1/2} |f(t)|\,dt,$$

which is some *fixed number independent* of n and T. Again we see that c_n tends to 0 like $1/T$, and so again we scale back up by T and consider

$$(\text{Scaled transform of periodized } f)(\frac{n}{T}) = Tc_n = \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)\,dt.$$

In the limit as $T \to \infty$ we replace n/T by s and consider

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt.$$

We're back to the same integral formula.

Fourier transform defined There you have it. We now *define* the *Fourier transform* of a function $f(t)$ to be

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt.$$

For now, just take this as a formal definition; we'll discuss later when such an integral exists. We assume that $f(t)$ is defined for all real numbers t. For any $s \in \mathbf{R}$, integrating $f(t)$ against $e^{-2\pi i s t}$ with respect to t produces a *complex valued* function of s, that is, the Fourier transform $\hat{f}(s)$ is a complex-valued function of $s \in \mathbf{R}$.

While the Fourier transform takes flight from the desire to find spectral information on a nonperiodic function, the extra complications and extra richness of what results will soon make it seem like we're in a much different world. The definition just given is a good one *because* of the richness and *despite* the complications. Periodic functions are great, but there's more bang than buzz in the world to analyze.

The spectrum for a periodic function is a discrete set of frequencies, possibly an infinite set (when there's a corner) but always a discrete set. In general, the Fourier transform of a nonperiodic signal produces a continuous spectrum, or a continuum of frequencies. If t has dimension 'time' then to make st dimensionless in the exponential $e^{-2\pi i s t}$ s must have dimension 1/time.

It may be that $\hat{f}(s)$ will be identically zero for $|s|$ sufficiently large, an important class of signals called *bandlimited*, or it may be that the nonzero values of $\hat{f}(s)$ extend to $\pm\infty$, or it may be that $\hat{f}(s)$ is zero for just a few values of s.

The Fourier transform analyzes a signal into its frequency components. We haven't yet considered how the corresponding synthesis goes. How can we get recover $f(t)$ in the time domain from $\hat{f}(s)$ in the frequency domain?

Recovering $f(t)$ from $\hat{f}(s)$ We can push the ideas on nonperiodic functions as limits of periodic functions a little further and discover how we might obtain $f(t)$ from its transform $\hat{f}(s)$. Again suppose $f(t)$ is zero outside some interval and periodize it to have (large) period T. We expand $f(t)$ in a Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}.$$

The Fourier *coefficients* can be written via the Fourier *transform* of f evaluated at the points $s_n = n/T$,

$$\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)\,dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi int/T} f(t)\,dt \\
&\qquad \text{(We can take the limits out to } \pm\infty \text{ since } f(t) \text{ will be zero outside of } [-T/2, T/2]) \\
&= \frac{1}{T}\hat{f}(\frac{n}{T}) \\
&= \frac{1}{T}\hat{f}(s_n).
\end{aligned}$$

Plug this in to the expression for $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t} \frac{1}{T}.$$

Now, the points $s_n = n/T$ are spaced $1/T$ apart, so we can think of $1/T$ as, say Δs, and the sum above as a Riemann sum approximating an integral

$$\sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t} \frac{1}{T} = \sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t} \Delta s \approx \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i st}\,ds.$$

The limits on the integral go from $-\infty$ to ∞ because the sum, and the points s_n, go from $-\infty$ to ∞. Thus as the period $T \to \infty$ we would expect to have

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i st}\,ds$$

and we have recovered $f(t)$ from $\hat{f}(s)$. We have found the *inverse Fourier transform.*

The inverse Fourier transform defined, and Fourier inversion, too The integral we've just come up with can stand on its own as a 'transform', and so we define the *inverse Fourier transform* of a function $g(s)$ to be

$$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi i st} g(s)\,ds \quad \text{(upside down hat – cute).}$$

Again, we're treating this formally for the moment, withholding a discussion of conditions under which the integral makes sense. In the same spirit, we've also produced the *Fourier inversion theorem*. That is

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} \hat{f}(s)\, ds.$$

Written very compactly,

$$(\hat{f})\check{} = f.$$

The inverse Fourier transform looks just like the Fourier transform except for the minus sign. Later we'll say more about the remarkable symmetry between the Fourier transform and its inverse.

By the way, we could have gone through the whole argument, above, starting with $\hat{f}$ as the basic function instead of f. If we did that we'd be led to the complementary result on Fourier inversion,

$$(\check{g})\hat{} = g.$$

A quick summary Let's summarize what we've done here, partly as a guide to what we'd like to do next. There's so much involved, all of importance, that it's hard to avoid saying everything at once. Realize that it will take some time before everything is in place.

- The Fourier transform of the signal $f(t)$ is

  $$\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t s}\, dt.$$

  This is a complex valued function of s.

- The domain of the Fourier transform is the set of $s \in \mathbf{R}$ where the integral defining it makes sense, and we'll have to understand that more thoroughly. One says that this is the frequency domain, and that the original signal is defined on the time domain (or the spatial domain, depending on the context). For a (nonperiodic) signal defined on the whole real line we generally do not have a discrete set of frequencies, as in the periodic case, but rather a *continuum* of frequencies.[2] (We still do call them 'frequencies', however.) The set of all frequencies is the *spectrum* of $f(t)$.

  - Not all frequencies need occur, *i.e.* $\hat{f}(s)$ might be zero for some values of s. Furthermore, it might be that there aren't any frequencies outside of a certain range, *i.e.*

    $$\hat{f}(s) \equiv 0 \quad \text{for} \quad |s| \quad \text{large}.$$

    These are called *bandlimited signals* and they are an important special class of signals. They come up in sampling theory.

- The inverse Fourier transform is

  $$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} g(s)\, ds$$

[2] A periodic function *does* have a Fourier transform, but it's a sum of δ functions. We'll have to do that, too, and it will take some effort.

Taken together, the Fourier transform and its inverse provide a way of passing between two (equivalent) representations of a signal via the Fourier inversion theorem:

$$(\hat{f})\check{} = f, \quad (\check{g})\hat{} = g.$$

Now remember that $\hat{f}(s)$ is a transformed, complex-valued function, and while it may be 'equivalent' to $f(t)$ it has very different properties. Is it really true that when $\hat{f}(s)$ exists we can just plug it into the formula for the inverse Fourier transform – which is also an improper integral that looks the same as the forward transform except for the minus sign – and really get back $f(t)$? Really? That's worth wondering about.

- The square magnitude $|\hat{f}(s)|^2$ (unintegrated) is called the *power spectrum* (especially in connection with its use in communications) or the *spectral power density* (especially in connection with its use in optics) or the *energy spectrum* (especially in every other connection).

  An important relation between the energy of the signal in the time domain and the energy spectrum in the frequency domain is given by Parseval's identity for Fourier transforms:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(s)|^2 \, ds.$$

  This is also a future attraction.

A warning on notation Depending on the operation to be performed, or on the context, it's often useful to have alternate notations for the Fourier transform. But here's a warning, which is the start of a complaint, which is the prelude to a full blown rant: Diddling with notation seems to be an unavoidable hassle in this subject. Flipping back and forth between a transform and it's inverse, naming the variables in the different domains, changing plus signs to minus signs, taking complex conjugates, these are all routine day-to-day operations, and they can cause endless muddles if you're not careful, and sometimes even if you are careful. You'll believe me when we have some examples, and you'll hear me complain about it frequently.

For example:

> If the signal is called f then one often uses the corresponding capital letter, like F, to denote the Fourier transform. Note, however, that one often uses different names for the variable, as in $f(x)$ (or $f(t)$) and $F(s)$.

And then there's this:

> Since taking the Fourier transform is an operation that is applied to a function to produce a new function, it's also sometimes convenient to indicate this by a kind of 'operational' notation. For example, it's common to write $\mathcal{F}f(s)$ for $\hat{f}(s)$, and so, to repeat he full definition
>
> $$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt.$$
>
> This is often the most unambiguous notation. Similarly, the operation of taking the inverse Fourier transform is then denoted by $\mathcal{F}^{-1}$, and so
>
> $$\mathcal{F}^{-1}g(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} g(s)\, ds$$
>
> I will probably slip into using these notations quite often.

Finally, a function and its Fourier transform are said to constitute a 'Fourier pair', to be explained more precisely later. There have been various notations devised to indicate this sibling relationship. One is

$$f(t) \rightleftharpoons F(s)$$

Bracewell advocated the use of

$$F(s) \supset f(t)$$

and Gray and Goodman also use it. I hate it, personally.

A warning on definitions Our definition of the Fourier transform is a standard one, but it's not the only one. It's a question of where to put the 2π: in the exponential, as we have done; or perhaps as a factor out front; or perhaps left out completely. There's also a question of which is the Fourier transform and which is the inverse, *i.e.*, which gets the minus sign in the exponential. I'll give you a handout called 'Chase the constants' that summarizes the many irritating variations. I only mention this now because when you're talking with a friend over drinks about the Fourier transform, be sure you both know which conventions are being followed. I'd hate to see that kind of misunderstanding get in the way of a beautiful friendship.

Examples

In one way, at least, our study of the Fourier transform will run the same course as your study of calculus. When you learned calculus it was necessary to learn the derivative and integral formulas for specific functions and types of functions (powers, exponentials, trig functions), and also to learn the general principles and rules of differentiation and integration that allow you to work with combinations of functions (product rule, chain rule, inverse functions). It will be the same thing for us now. We'll need to have a storehouse of specific functions and their transforms that we can call on, and we'll need to develop general principles and results on how the Fourier transform operates.

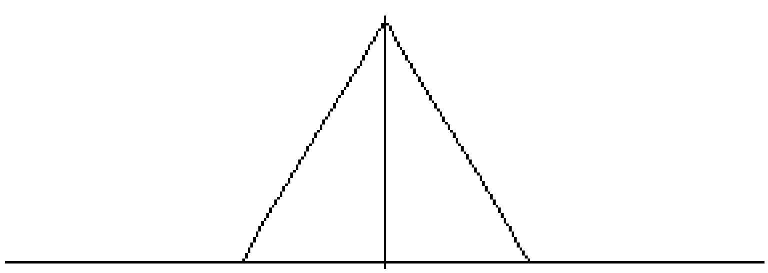
We've already seen the example

$$\widehat{\Pi} = \mathrm{sinc}$$

Let's do a few more examples.

The triangle function Consider next the 'triangle function', defined by

$$\Lambda(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

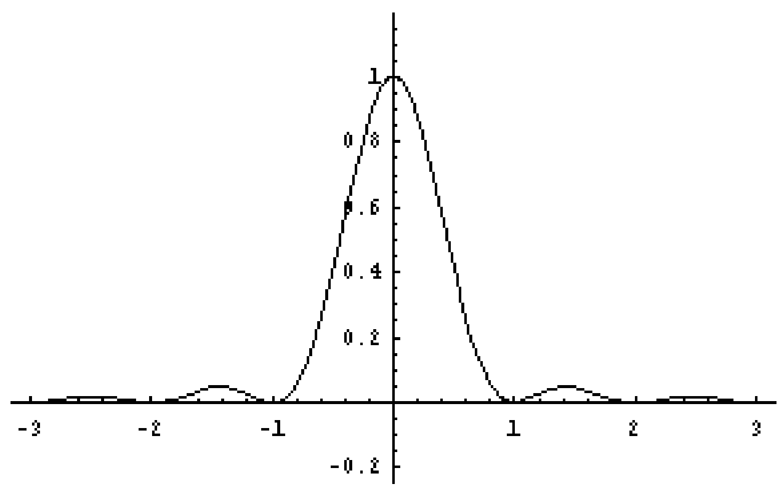


For the Fourier transform we compute (using integration by parts, and the factoring trick for the sine function):

$$
\begin{aligned}
\hat{\Lambda}(s) &= \int_{-\infty}^{\infty} \Lambda(x) e^{-2\pi i s x}\, dx = \int_{-1}^{0} (1+x) e^{-2\pi i s x}\, dx + \int_{0}^{1} (1-x) e^{-2\pi i s x}\, dx \\
&= \left(\frac{1+2i\pi s}{4\pi^2 s^2} - \frac{e^{2\pi i s}}{4\pi^2 s^2} \right) - \left(\frac{2i\pi s - 1}{4\pi^2 s^2} + \frac{e^{-2\pi i s}}{4\pi^2 s^2} \right) \\
&= -\frac{e^{-2\pi i s}(e^{2\pi i s} - 1)^2}{4\pi^2 s^2} \\
&= -\frac{e^{-2\pi i s}(e^{\pi i s}(e^{\pi i s} - e^{-\pi i s}))^2}{4\pi^2 s^2} \\
&= -\frac{e^{-2\pi i s} e^{2\pi i s} (2i)^2 \sin^2 \pi s}{4\pi^2 s^2} \\
&= \left(\frac{\sin \pi s}{\pi s} \right)^2 \\
&= \operatorname{sinc}^2 s.
\end{aligned}
$$

It's no accident that the Fourier transform of the triangle function turns out to be the square of the Fourier transform of the rect function. It has to do with convolution, an operation we have seen for Fourier series and will see anew for Fourier transforms.

The graph of $\operatorname{sinc}^2 s$ looks like:

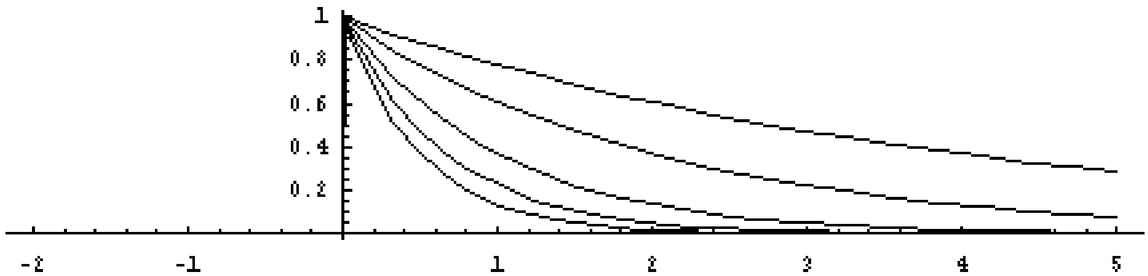


The exponential decay Another commonly occuring function is the (one-sided) exponential decay, defined by

$$
f(t) = \begin{cases} 0 & t \le 0, \\ e^{-at} & t > 0, \end{cases}
$$

where a is a positive constant. This function models a signal that is zero, switched on, and then decays exponentially.

Here are graphs for $a = 2$, 1.5, 1.0, 0.5 and 0.25.



Which is which? If you can't say, see the Appendix.

Back to the exponential decay, we can calculate its Fourier transform directly:

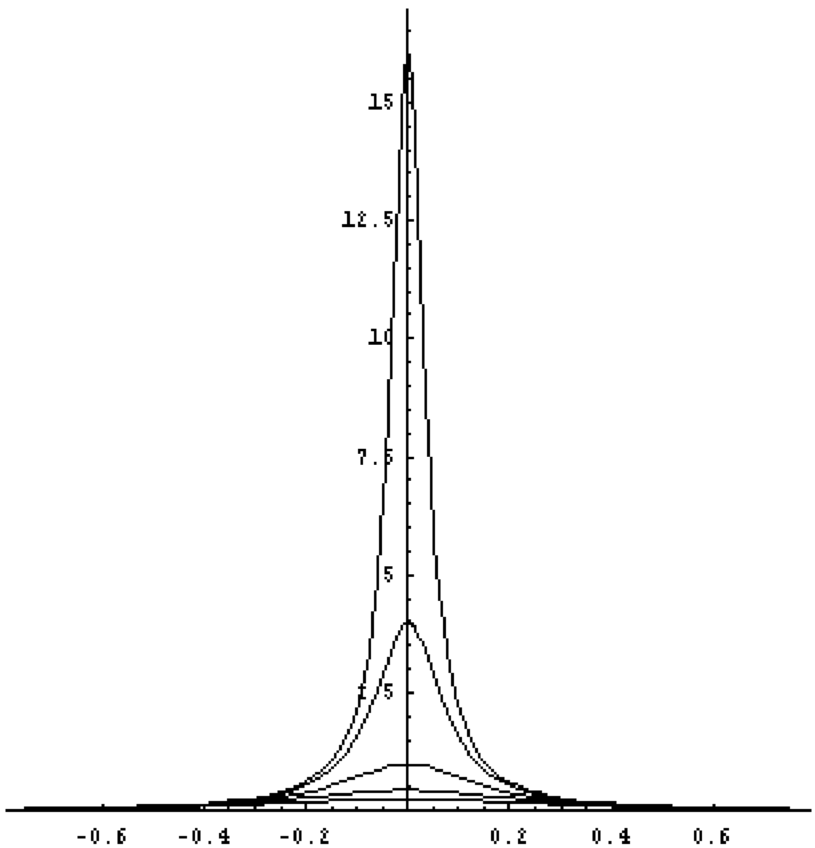
$$\begin{aligned}
\hat{f}(s) &= \int_0^{\infty} e^{-2\pi ist} e^{-at}\, dt \\
&= \int_0^{\infty} e^{-2\pi ist-at}\, dt \\
&= \int_0^{\infty} e^{(-2\pi is-a)t}\, dt \\
&= \left[\frac{e^{(-2\pi is-a)t}}{-2\pi is-a}\right]_{t=0}^{t=\infty} \\
&= \frac{1}{2\pi is+a}
\end{aligned}$$

In this case, unlike the results for the rect function and the triangle function, the Fourier transform is complex. The fact that $\widehat{\Pi}(s)$ and $\hat{\Lambda}(s)$ are real is because $\Pi(x)$ and $\Lambda(x)$ are even functions. (We'll go over this later.) There is no symmetry for the exponential decay.

The power spectrum of the exponential decay is

$$|\hat{f}(s)|^2 = \frac{1}{|2\pi is+a|^2} = \frac{1}{a^2+4\pi^2s^2}.$$

Here are graphs of this function for the same values of a as in the graphs of the exponential decay function.



Which is which? You'll soon learn to spot that immediately, relative to the picture in the time domain, and it's an important issue.

The shape of the Fourier transform is that of a 'bell curve', though this is *not* a 'Gaussian' (which we'll discuss later). The curve is known as a 'Lorenz profile' and comes up in analyzing the transition probabilities and lifetime of the excited state in atoms.

Appendix: How does the graph of $f(ax)$ compare with the graph of $f(x)$?

Let me remind you of some elementary lore on scaling the independent variable in a function and how that affects its graph. The question is how the graph of $f(ax)$ compares with the graph of $f(x)$ when $a < 1$ and when $a > 1$. (I'm talking about any generic function $f(x)$ here.) This is very simple, especially compared to what we've done and what we're going to do, but you'll want it at your fingertips and *everyone* has to think about it for a few seconds. Here's how to spend those few seconds.

Take, for example, the graph of $f(2x)$. The graph of $f(2x)$, compared with the graph of $f(x)$, is squeezed: Think about what happens when you plot the graph of $f(2x)$ over, say $-1 \le x \le 1$. When x goes from -1 to 1, $2x$ goes from -2 to 2, so while you're plotting $f(2x)$ over the interval from -1 to 1 you have to compute the values of $f(x)$ from -2 to 2. That's more of the function in less space, so the graph of $f(2x)$ is a squeezed version of the graph of $f(x)$. Clear?

Similar reasoning shows that the graph of $f(x/2)$ is stretched. If x goes from -1 to 1 then $x/2$ goes from $-1/2$ to $1/2$, so while you're plotting $f(x/2)$ over the interval -1 to 1 you have to compute the values of $f(x)$ from $-1/2$ to $1/2$. That's less of the function in more space, so the graph of $f(x/2)$ is a stretched version of the graph of $f(x)$.