EE 340
Electromagnetics
Lab

Problem Session #3

Prob 3.15: Determine the gradient of the following scalar fields:

$$\boxed{\text{Since } \nabla U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z}$$

a) $U = 4xz^2 + 3yz$

$$\vec{\nabla} U = 4z^2\,\bar{a}_x + 3z\,\bar{a}_y + (8xz + 3y)\bar{a}_z$$

b) $V = e^{(2x+3y)}\cos 5z$

$$\vec{\nabla} V = e^{(2x+3y)}[2\cos 5z\,\bar{a}_x + 3\cos 5z\,\bar{a}_y - 5\sin 5z\,\bar{a}_z]$$

c) $W = 2\rho(z^2+1)\cos\phi$

$$\vec{\nabla} W = 2(z^2+1)\cos\phi\,\bar{a}_\rho - 2(z^2+1)\sin\phi\,\bar{a}_\phi + 4\rho z\cos\phi\,\bar{a}_z$$

d) $T = 5\rho e^{-2z}\sin\phi$

$$\vec{\nabla} T = 5e^{-2z}[\sin\phi\,\bar{a}_\rho + \cos\phi\,\bar{a}_\phi - 2\rho\sin\phi\,\bar{a}_z]$$

e) $H = r^2\cos\theta\cos\phi$

$$\vec{\nabla} H = 2r\cos\theta\cos\phi\,\bar{a}_r - r\sin\theta\cos\phi\,\bar{a}_\theta - r\cos\theta\sin\phi\,\bar{a}_\phi$$

f) $Q = (\sin\theta\sin\phi)/r^3$

$$\vec{\nabla} Q = \frac{1}{r^4}[-3\sin\theta\sin\phi\,\bar{a}_r + \cos\theta\sin\phi\,\bar{a}_\theta + \cos\phi\,\bar{a}_\phi]$$

Prob 3.18: Find the divergence and curl of the following vectors:

Since $\overline{\nabla} = \frac{\partial}{\partial x}\overline{a_x} + \frac{\partial}{\partial y}\overline{a_y} + \frac{\partial}{\partial z}\overline{a_z}$

a) $A = \underbrace{e^{xy}}_{A_x}\,\overline{a}_x + \underbrace{\sin xy}_{A_y}\,\overline{a}_y + \underbrace{\cos^2 xz}_{A_z}\,\overline{a}_z$

$$\text{Divergence of } A = \vec{\nabla}\bullet\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\Rightarrow \quad \vec{\nabla}\bullet\vec{A} = ye^{xy} + x\cos xy - 2x\cos(xz)\sin(xz)$$

and Curl is; $\overline{\nabla}\times\overline{A} = \left[\frac{\delta A_z}{\delta y} - \frac{\delta A_y}{\delta z}\right]\overline{a}_x - \left[\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta z}\right]\overline{a}_y + \left[\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right]\overline{a}_z$

$+\left[\frac{\delta A_x}{\delta z} - \frac{\delta A_z}{\delta x}\right]\overline{a}_y$

$$\vec{\nabla}\times\vec{A} = (0-0)a_x + (0+2z\cos(xz)\sin(xz))a_y + (y\cos(xy) - xe^{xy})a_z$$

$$= 2z\cos(xz)\sin(xz)\,\overline{a}_y + (y\cos(xy) - xe^{xy})\overline{a}_z$$

Prob 3:18(b): (Pg 77)

$$\overline{\nabla}\times\overline{B} = \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\delta}{\delta\rho} & \frac{\delta}{\delta\phi} & \frac{\delta}{\delta z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}$$

$$\boxed{\overline{\nabla}.\overline{A} = \frac{1}{\rho}\frac{\delta}{\delta\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\delta A_\phi}{\delta\phi} + \frac{\delta A_z}{\delta z}}$$

b) $B = \rho z^2 \cos\phi\,\overline{a}_\rho + z\sin^2\phi\,\overline{a}_z$

$$\vec{\nabla}\bullet\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2 z^2\cos\phi) + 0 + \sin^2\phi$$

$$= 2z^2\cos\phi + \sin^2\phi$$

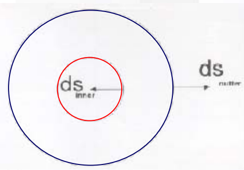
$$\text{and} \quad \vec{\nabla}\times\vec{B} = \frac{z\sin\phi}{\rho}\,\overline{a}_\rho + 2\rho z\cos\phi\,\overline{a}_\phi + z^2\sin\phi\,\overline{a}_z$$

Prob 3.18(c): $C = r\cos\theta\, a_r - \frac{1}{r}\sin\theta\, a_\theta + 2r^2\sin\theta\, a_\phi$

$$\vec{\nabla}\bullet\vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-1}{r}\sin^2\theta\right) + 0 = 3\cos\theta - \frac{2\cos\theta}{r^2}$$

$$and \quad \vec{\nabla}\times\vec{C} = 4r\cos\theta\,\bar{a}_r - 6r\sin\theta\,\bar{a}_\theta + \sin\theta\,\bar{a}_\phi$$

Prob 3.30(a): Given that $E = \frac{1}{r^4}\sin^2\phi\,\bar{a}_r$, Evaluate the following over the region between the spherical surfaces r =2 and r = 4.



a) $\int_s E\bullet dS \begin{cases} dS_{outter} = r^2\sin\theta\, d\theta\, d\phi\,\bar{a}_r \\ dS_{inner} = -r^2\sin\theta\, d\theta\, d\phi\,\bar{a}_r \end{cases}$

$$\oint_s E\bullet dS = \int_{inner} E\bullet dS_{inner} + \int_{outter} E\bullet dS_{outter} = \frac{-r_{in}^2}{r_{in}^4}\int_0^{2\pi}\int_0^{\pi}\sin^2\phi\sin\theta\, d\theta\, d\phi + \frac{r_{out}^2}{r_{out}^4}\int_0^{2\pi}\int_0^{\pi}\sin^2\phi\sin\theta\, d\theta\, d\phi$$

$$= \left(\frac{1}{4^2} - \frac{1}{2^2}\right)\int_0^{2\pi}\sin^2\phi\, d\phi\int_0^{\pi}\sin\theta\, d\theta = \left(\frac{1}{4}\cdot\frac{-3}{4}\right)\cdot(\pi)(2) = \frac{-3\pi}{8}$$

Prob 3.30(b):

$$\int_V (\nabla\bullet E)\, dv$$

$$Since \quad \vec{\nabla}\bullet\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\sin^2\phi\right) = \frac{-2}{r^5}\sin^2\phi$$

$$\int\vec{\nabla}\bullet\vec{E}\, dv = -2\int_0^{2\pi}\int_0^{\pi}\int_2^4\frac{1}{r^3}\sin^2\phi\sin\theta\, dr\, d\theta\, d\phi$$

$$= -2(\pi)(2)\cdot\frac{-1}{2}\left(\frac{1}{4^2} - \frac{1}{2^2}\right) = -2\pi\cdot\frac{+3}{16} = \frac{-3\pi}{8}$$

Prob 3.33: Calculate the total outward flux of vector $F = \rho^2 \sin\phi \, a_\rho + z\cos\phi \, a_\phi + \rho z \, a_z$ through the hollow cylinder defined by $2 \le \rho \le 3, 0 \le z \le 5$

$$\psi_{inner} = -\int_0^5 \int_0^{2\pi} \rho^3 \sin\phi \, d\phi \, dz = 0 \ (Since \ \ 0 \le \phi \le 2\pi)$$

(at $\rho = 2$)

$$\psi_{outter} = \int_0^5 \int_0^{2\pi} \rho^3 \sin\phi \, d\phi \, dz = 0 \ (Since \ \ 0 \le \phi \le 2\pi)$$

(at $\rho = 3$)

$$So, \ \psi_{total} = \frac{190\pi}{3}$$

But the total outward flux can be found as the following:

$$\int_V (\nabla \bullet F) dv \ \ where, \ dv = \rho d\rho d\phi dz$$ (divergence theorem)

$$\Rightarrow \int_0^{2\pi}\int_0^5\int_2^3 (\vec{\nabla} \bullet \vec{F}) \rho d\rho d\phi dz = \frac{190\pi}{3}$$

Prob 3.39: Given the vector field

$$R = (2x^2 y + yz)\bar{a}_x + (xy^2 - xz^3)\bar{a}_y + (\mathbf{c}xyz - 2x^2 y^2)\bar{a}_z$$

Determine the value of **c** for **R** to be solenoid.

Sol. (Divergence of closes sur int. Less)

To be solenoid, that means $\vec{\nabla} \bullet \vec{R} = 0$

$$\Rightarrow \vec{\nabla} \bullet \vec{R} = 4xy + 2xy + cxy = 0$$

$$\Rightarrow C = -6$$

Problem # 3.40 If the vector field (Pg 87)

$$T = (\alpha xy + \beta z^3)\bar{a}_x + (3x^2 - \gamma z)\bar{a}_y + (3xz^2 - y)\bar{a}_z$$

is irrotational, determine

$\alpha, \beta, and \gamma$. Find $\nabla \bullet T$ at $(2, -1, 0)$

Sol. Since T is irrotational, that means $\vec{\nabla} \times \vec{T} = 0$ (curl free $\to$ "g")

$$\vec{\nabla} \times \vec{T} = (-1 + \gamma)\bar{a}_x + (3\beta z^2 - 3z^2)\bar{a}_y + (6x - \alpha x)\bar{a}_z = 0$$

$$\therefore \ \alpha = 6 \ , \ \beta = 1 \ and \ \ \gamma = 1$$

as, $$\vec{\nabla} \times \bar{T} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ T_x & T_y & T_z \end{vmatrix}$$

$\boxed{\text{Again } \vec{\nabla} \cdot \bar{T} \Rightarrow \text{Solve it.}}$

(See problem 3.18(a))