

# EE 340 Electromagnetics Lab

## Problem Session #3

**Prob 3.15:** Determine the gradient of the following scalar fields:

$$\text{Since } \vec{\nabla} U = \frac{\partial U}{\partial x} \vec{a}_x + \frac{\partial U}{\partial y} \vec{a}_y + \frac{\partial U}{\partial z} \vec{a}_z$$

a)  $U = 4xz^2 + 3yz$

$$\vec{\nabla} U = 4z^2 \vec{a}_x + 3z \vec{a}_y + (8xz + 3y) \vec{a}_z$$

b)  $V = e^{(2x+3y)} \cos 5z$

$$\vec{\nabla} V = e^{(2x+3y)} [2 \cos 5z \vec{a}_x + 3 \cos 5z \vec{a}_y - 5 \sin 5z \vec{a}_z]$$

c)  $W = 2\rho(z^2 + 1) \cos \phi$

$$\vec{\nabla} W = 2(z^2 + 1) \cos \phi \vec{a}_\rho - 2(z^2 + 1) \sin \phi \vec{a}_\phi + 4\rho z \cos \phi \vec{a}_z$$

d)  $T = 5\rho e^{-2z} \sin \phi$

$$\vec{\nabla} T = 5e^{-2z} [\sin \phi \vec{a}_\rho + \cos \phi \vec{a}_\phi - 2\rho \sin \phi \vec{a}_z]$$

e)  $H = r^2 \cos \theta \cos \phi$

$$\vec{\nabla} H = 2r \cos \theta \cos \phi \vec{a}_r - r \sin \theta \cos \phi \vec{a}_\theta - r \cos \theta \sin \phi \vec{a}_\phi$$

f)  $Q = (\sin \theta \sin \phi) / r^3$

$$\vec{\nabla} Q = \frac{1}{r^4} [-3 \sin \theta \sin \phi \vec{a}_r + \cos \theta \sin \phi \vec{a}_\theta + \cos \phi \vec{a}_\phi]$$

**Prob 3.18:** Find the divergence and curl of the following vectors:

$$\text{Since } \vec{\nabla} = \frac{\partial}{\partial x} \vec{a}_x + \frac{\partial}{\partial y} \vec{a}_y + \frac{\partial}{\partial z} \vec{a}_z$$

a)  $A = \frac{e^{-xy}}{Ax} \vec{a}_x + \frac{\sin xy}{Ay} \vec{a}_y + \frac{\cos^2 xz}{Az} \vec{a}_z$

$$\text{Divergence of } A = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{A} = ye^{-xy} + x \cos xy - 2x \cos(xz) \sin(xz)$$

and curl is;  $\vec{\nabla} \times \vec{A} = \left[ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] \vec{a}_x - \left[ \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right] \vec{a}_y + \left[ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right] \vec{a}_z$

$$\vec{\nabla} \times \vec{A} = (0-0) \vec{a}_x + (0+2z \cos(xz) \sin(xz)) \vec{a}_y + (y \cos(xy) - xe^{-xy}) \vec{a}_z$$

$$= 2z \cos(xz) \sin(xz) \vec{a}_y + (y \cos(xy) - xe^{-xy}) \vec{a}_z$$

**Prob 3.18(b):**

(Cylindrical)  $\vec{\nabla} \times \vec{B} = \begin{vmatrix} \vec{a}_\rho & \rho \vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

b)  $B = \rho z^2 \cos \phi \vec{a}_\rho + z \sin^2 \phi \vec{a}_z$

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 z^2 \cos \phi) + 0 + \sin^2 \phi$$

$$= 2z^2 \cos \phi + \sin^2 \phi$$

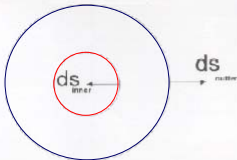
and  $\vec{\nabla} \times \vec{B} = \frac{z \sin \phi}{\rho} \vec{a}_\rho + 2\rho z \cos \phi \vec{a}_\phi + z^2 \sin \phi \vec{a}_z$

**Prob 3.18(c):**  $C = r \cos \theta \bar{a}_r - \frac{1}{r} \sin \theta \bar{a}_\theta + 2r^2 \sin \theta \bar{a}_\phi$

$$\vec{\nabla} \cdot \vec{C} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( -\frac{1}{r} \sin^2 \theta \right) + 0 = 3 \cos \theta - \frac{2 \cos \theta}{r^2}$$

and  $\vec{\nabla} \times \vec{C} = 4r \cos \theta \bar{a}_r - 6r \sin \theta \bar{a}_\theta + \sin \theta \bar{a}_\phi$

**Prob 3.30(a):** Given that  $E = \frac{1}{r^4} \sin^2 \phi \bar{a}_r$ , Evaluate the following over the region between the spherical surfaces  $r=2$  and  $r=4$ .



a)  $\int_s E \cdot dS \begin{cases} dS_{outer} = r^2 \sin \theta d\theta d\phi \bar{a}_r \\ dS_{inner} = -r^2 \sin \theta d\theta d\phi \bar{a}_r \end{cases}$

$$\oint_s E \cdot dS = \int_{inner} E \cdot dS_{inner} + \int_{outer} E \cdot dS_{outer} = \frac{-r_{in}^2}{r_m^4} \int_0^{2\pi} \int_0^\pi \sin^2 \phi \sin \theta d\theta d\phi + \frac{r_{out}^2}{r_m^4} \int_0^{2\pi} \int_0^\pi \sin^2 \phi \sin \theta d\theta d\phi$$

$$= \left( \frac{1}{4^4} - \frac{1}{2^4} \right) \int_0^{2\pi} \sin^2 \phi d\phi \int_0^\pi \sin \theta d\theta = \left( \frac{1}{4} - \frac{1}{8} \right) (\pi) (2) = \frac{-2\pi}{8}$$

**Prob 3.30(b):**

$$\int_V (\nabla \cdot E) dv$$

Since  $\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r^2} \sin^2 \phi \right) = \frac{-2}{r^5} \sin^2 \phi$

$$\int \vec{\nabla} \cdot \vec{E} dv = -2 \int_0^{2\pi} \int_0^\pi \int_2^4 \frac{1}{r^3} \sin^2 \phi \sin \theta dr d\theta d\phi$$

$$= -2 (\pi) (2) \cdot \frac{1}{2} \left( \frac{1}{4^2} - \frac{1}{2^2} \right) = -2\pi \cdot \frac{+3}{16} = \frac{-3\pi}{8}$$

**Prob 3.33:** Calculate the total outward flux of vector  $F = \rho^2 \sin \phi a_\rho + z \cos \phi a_\phi + \rho z a_z$  through the hollow cylinder defined by  $2 \leq \rho \leq 3, 0 \leq z \leq 5$

$$\psi_{inner} = - \int_0^{5} \int_0^{2\pi} \rho^3 \sin \phi d\phi dz = 0 \quad (\text{Since } 0 \leq \phi \leq 2\pi) \quad (\text{at } \rho=2)$$

$$\psi_{outer} = \int_0^{5} \int_0^{2\pi} \rho^3 \sin \phi d\phi dz = 0 \quad (\text{Since } 0 \leq \phi \leq 2\pi) \quad (\text{at } \rho=3)$$

$$\text{So, } \psi_{total} = \frac{190\pi}{3}$$

But the total outward flux can be found as the following:

$$\int_V (\nabla \cdot F) dv \quad \text{where, } dv = \rho d\rho d\phi dz \quad (\text{divergence theorem})$$

$$\Rightarrow \int_0^5 \int_0^{2\pi} \int_2^3 (\nabla \cdot \vec{F}) \rho d\rho d\phi dz = \frac{190\pi}{3}$$

**Prob 3.39:** Given the vector field  $R = (2x^2y + yz)\vec{a}_x + (xy^2 - xz^3)\vec{a}_y + (cxyz - 2x^2y^2)\vec{a}_z$ . Determine the value of  $c$  for  $R$  to be solenoid.

**Sol.** (Divergence is closed surface int. less)  
To be solenoid, that means  $\nabla \cdot \vec{R} = 0$   
 $\Rightarrow \vec{\nabla} \cdot \vec{R} = 4xy + 2xy + cxy = 0$   
 $\Rightarrow C = -6$

**Problem # 3.40** If the vector field  $T = (\alpha xy + \beta z^3)\vec{a}_x + (3x^2 - \gamma z)\vec{a}_y + (3xz^2 - y)\vec{a}_z$  is irrotational, determine  $\alpha, \beta$ , and  $\gamma$ . Find  $\nabla \cdot T$  at  $(2, -1, 0)$

**Sol.** Since  $T$  is irrotational, that means  $\vec{\nabla} \times \vec{T} = 0$   
(curl free  $\rightarrow \vec{\nabla} \times \vec{T} = 0$ )

$$\vec{\nabla} \times \vec{T} = (-1 + \gamma)\vec{a}_x + (3\beta z^2 - 3z^2)\vec{a}_y + (6x - \alpha x)\vec{a}_z = 0$$

$$\therefore \alpha = 6, \beta = 1 \text{ and } \gamma = 1$$

(Pg 87)

$$\text{as, } \vec{\nabla} \times \vec{T} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T_x & T_y & T_z \end{vmatrix}$$

Again  $\vec{\nabla} \cdot \vec{T} \Rightarrow$  Solve it. (See problem 3.18(a))