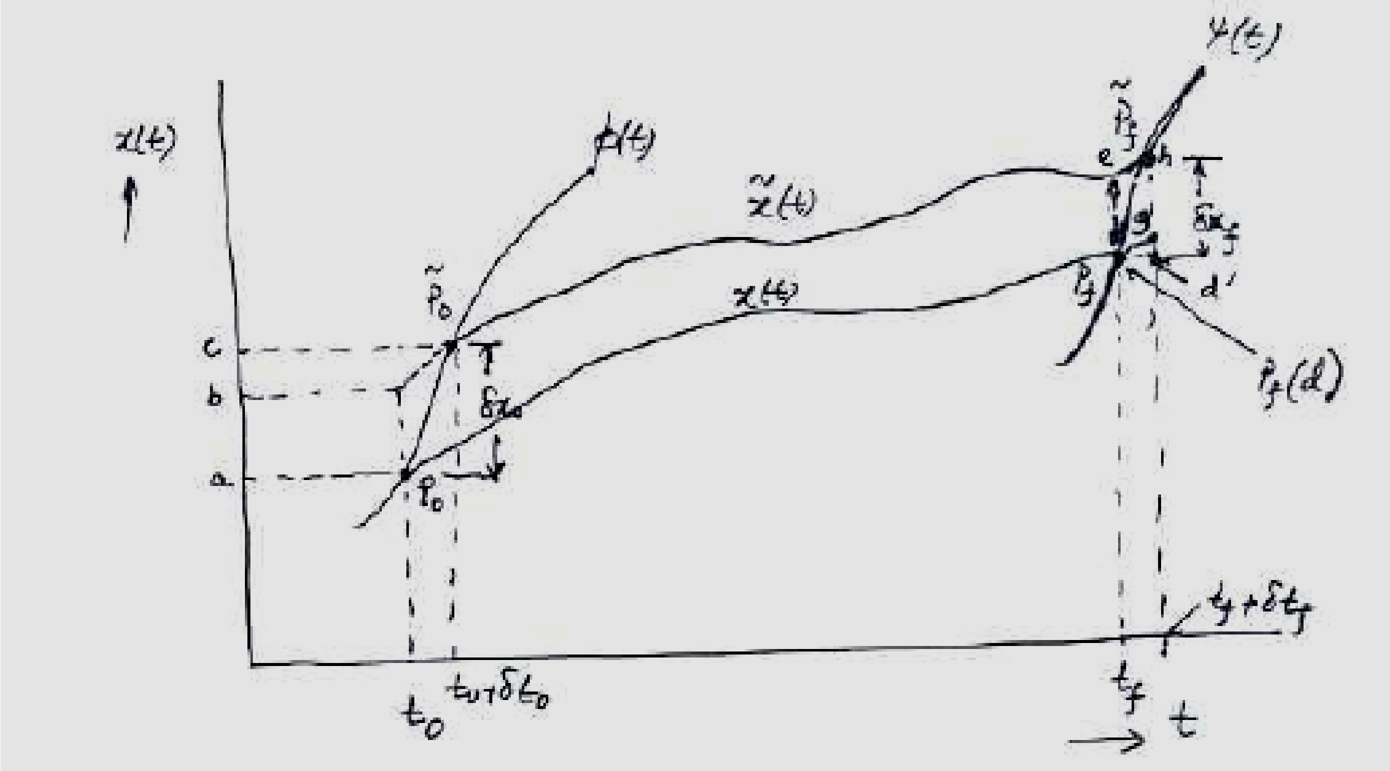
General Variable End Point Problem

Find the curve x(t) which minimizes the functional

$$J = \int_{t_o}^{t_f} L(x, \dot{x}, t)\,dt \qquad (1)$$

The end points of x(t) are allowed to travel along specified curves φ(t) and ψ(t) as shown in the figure.



Consider two neighboring curves x(t) and $\tilde{x}(t)$. Let P_o and $\tilde{P}_o$ be the left end points of x(t) and $\tilde{x}(t)$, respectively. Similarly, P_f and $\tilde{P}_f$ are the right end points.

Coordinates of	P_o are	(t_o, x_o)
	$\tilde{P}_o$	$(t_o + \delta t_o, x_o + \delta x_o)$
	P_f	(t_f, x_f)
	$\tilde{P}_f$	$(t_f + \delta t_f, x_f + \delta x_f)$

Assume that the curves x(t) and $\tilde{x}(t)$ are defined over the common extended interval $[t_o, t_f + \delta t_f]$. Extrapolate x(t) and $\tilde{x}(t)$ appropriately.

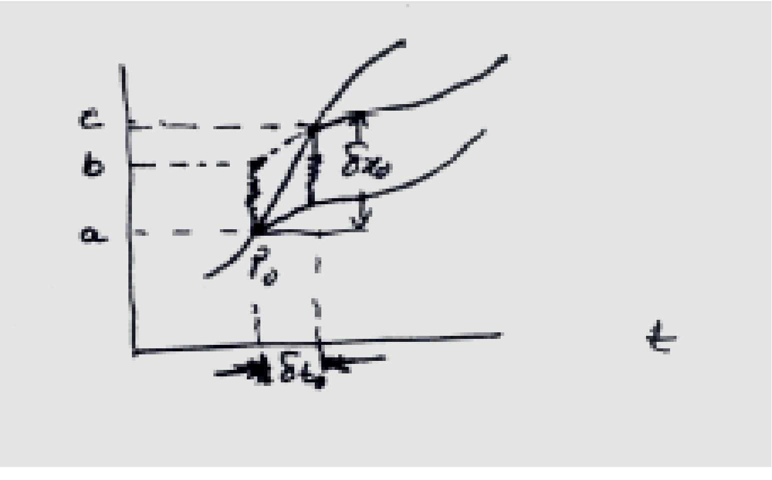
Let, $$\tilde{x}(t) = x(t) + \varepsilon\eta(t)$$

Then,

$$\varepsilon\eta(t) = \tilde{x}(t) - x(t)$$

$$\varepsilon\eta(t_o) = \tilde{x}(t_o) - x(t_o) = \text{ ab from the figure} \tag{2}$$

$$= \delta x_o - \dot{x}(t_o)\delta t_o$$



Similarly,

$$\varepsilon\eta(t_f) = \tilde{x}(t_f) - x(t_f)$$
$$= \text{de}$$
$$= \text{hd}'\text{-gd}' \text{ (former diagram)}$$
$$= \delta x_f - \dot{x}(t_f)\delta t_f \qquad (3)$$

The variation of J is,

$$\delta J = J[\tilde{x}(t)] - J[x(t)]$$
$$= \int_{t_o+\delta t_o}^{t_f+\delta t_f} L(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t)dt - \int_{t_o}^{t_f} L(x,\dot{x},t)dt$$

Or,

$$\delta J = \int_{t_o}^{t_f} \{L(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t) - L(x,\dot{x},t)\}dt$$
$$+ \int_{t_f}^{t_f+\delta t_f} L(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t)dt \ - \int_{t_o}^{t_o+\delta t_o} L(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t)dt \qquad (4)$$

Expand the integrands in each of the three integrals in Taylor series around x(t) to yield,

$$
\begin{aligned}
\delta J = & \int_{t_o}^{t_f} \varepsilon\{L_x \eta + L_{\dot{x}} \dot{\eta}\} dt + o(\varepsilon^2) \\
& + \int_{t_f}^{t_f + \delta t_f} \{L(x, \dot{x}, t) + L_x \varepsilon \eta + L_{\dot{x}} \varepsilon \dot{\eta} + o(\varepsilon^2)\} dt \\
& - \int_{t_o}^{t_o + \delta t_o} \{L(x, \dot{x}, t) + L_x \varepsilon \eta + L_{\dot{x}} \varepsilon \dot{\eta} + o(\varepsilon^2)\} dt
\end{aligned}
\tag{5}
$$

1. Neglect term $o(\varepsilon^2)$ in each of the three integrals, since it represents higher order terms involving $\varepsilon^2, \varepsilon^3$, etc.
2. In the second and third integrals we can ignore $L_x \varepsilon \eta$ and $L_{\dot{x}} \varepsilon \dot{\eta}$ because they involve product terms $\delta t_o, \delta t_f, \delta x_o$ and δx_f

So, we get,

$$
\begin{aligned}
\delta J = & \int_{t_o}^{t_f} \varepsilon\{L_x \eta + L_{\dot{x}} \dot{\eta}\} dt \quad + L(x, \dot{x}, t)\Big|_{t_f} \delta t_f \\
& - L(x, \dot{x}, t)\Big|_{t_o} \delta t_o
\end{aligned}
\tag{6}
$$

Integrate the second term inside the integral by parts to give

$$L_{\dot{x}}\varepsilon\eta\Big|_{t_o}^{t_f} - \int_{t_o}^{t_f}\varepsilon\eta\frac{d}{dt}(L_{\dot{x}})dt$$

$$= L_{\dot{x}}\Big|_{t_f}\varepsilon\eta(t_f) - L_{\dot{x}}\Big|_{t_o}\varepsilon\eta(t_o) - \int_{t_o}^{t_f}\varepsilon\eta\frac{d}{dt}L_{\dot{x}}dt \qquad (7)$$

Substituting this in the expression for δJ in (6),

$$\delta J = \int_{t_o}^{t_f}(L_x - \frac{d}{dt}L_{\dot{x}})\varepsilon\eta dt \quad + L(x,\dot{x},t)\Big|_{t_f}\delta t_f \quad - L(x,\dot{x},t)\Big|_{t_o}\delta t_o$$

$$+ L_{\dot{x}}(x,\dot{x},t)\Big|_{t_f}\varepsilon\eta(t_f) - L_{\dot{x}}(x,\dot{x},t)\Big|_{t_o}\varepsilon\eta(t_o) \qquad (8)$$

From (2) and (3), substitute for $\varepsilon\eta(t_o)$ and $\varepsilon\eta(t_f)$ in terms of δx_o and δx_f to get,

$$\delta J = \int_{t_o}^{t_f}(L_x - \frac{d}{dt}L_{\dot{x}})\varepsilon\eta dt \quad + L(x,\dot{x},t)\Big|_{t_f}\delta t_f \quad - L(x,\dot{x},t)\Big|_{t_o}\delta t_o$$

$$+ L_{\dot{x}}(x,\dot{x},t)\Big|_{t_f}(\delta x_f - \dot{x}(t_f)\delta t_f) - L_{\dot{x}}(x,\dot{x},t)\Big|_{t_o}(\delta x_o - \dot{x}(t_o)\delta t_o) \qquad (9)$$

$$= \int_{t_o}^{t_f}(L_x - \frac{d}{dt}L_{\dot{x}})\varepsilon\eta dt \quad + L_{\dot{x}}\delta x\Big|_{t_o}^{t_f} + (L - L_{\dot{x}}\dot{x})\delta t\Big|_{t_o}^{t_f}$$

For J to be extremized by x(t), δJ must be = 0. Since η(t) is arbitrary and so also δx and δt, the following conditions must hold:

$$L_x - \frac{d}{dt} L_{\dot{x}} = 0 \qquad (10)$$

⇨ *The Euler- Lagrange equation.*

$$L_{\dot{x}}\, \delta x \Big|_{t_o}^{t_f} + (L - L_{\dot{x}} \dot{x}) \delta t \Big|_{t_o}^{t_f} = 0 \qquad (11)$$

⇨ *The transversality conditions.*

Note that so far no specific use has been made of φ(t) and ψ(t). Consequently, the preceding analysis is valid for any arbitrary movement of the end points.

If φ(t) and ψ(t) are specified as the paths for the end points, we will get the following relationships:

$$\begin{aligned} \delta x_o &= \dot{\phi}(t_o) \delta t_o \\ \delta x_f &= \dot{\psi}(t_f) \delta t_f \end{aligned} \qquad (12)$$

The transversality conditions become,

$$L_{\dot{x}}\Big|_{t_f} \dot{\psi}(t_f)\delta t_f - L_{\dot{x}}\Big|_{t_o} \dot{\phi}(t_o)\delta t_o + (L - L_{\dot{x}}\dot{x})\Big|_{t_f} \delta t_f - (L - L_{\dot{x}}\dot{x})\Big|_{t_o} \delta t_o = 0$$

Or,

$$(L_{\dot{x}}\dot{\psi} + L - L_{\dot{x}}\dot{x})\Big|_{t_f} \delta t_f - (L_{\dot{x}}\dot{\phi} + L - L_{\dot{x}}\dot{x})\Big|_{t_o} \delta t_o = 0$$

If δt_f and δt_o are both arbitrary, the transversality conditions become,

$$[L_{\dot{x}}(\dot{\psi} - \dot{x}) + L]\Big|_{t_f} = 0 \qquad (12)$$

$$[L_{\dot{x}}(\dot{\phi} - \dot{x}) + L]\Big|_{t_o} = 0 \qquad (13)$$

For solving the variable end point problem, the procedure is:

1. Solve the Euler equation and get a general solution
2. Then apply the appropriate transversality conditions to evaluate the arbitrary constants.