Home Work #2: State Variable Methods

E3.9 Analyzing the two nodes using Kirchoff's current law yields

$$\dot{x}_1 = \frac{1}{2}x_2 - x_1$$
$$\dot{x}_2 = -\dot{x}_1 - x_2 .$$

In state-variable form we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & -\frac{3}{2} \end{bmatrix} \mathbf{x} , \quad y = [1 \quad -\frac{3}{2}]\mathbf{x} .$$

The characteristic equation is

$$s^2 + \frac{5}{2}s + 1 = (s+2)(s+\frac{1}{2}) = 0 .$$

E3.18 The governing equations of motion are

$$Ri_1 + L_1\frac{di_1}{dt} + v = v_a$$
$$L_2\frac{di_2}{dt} + v = v_b$$
$$i_L = i_1 + i_2 = C\frac{dv}{dt} .$$

Let $x_1 = i_1, x_2 = i_2, x_3 = v, u = v_a$ and $u_2 = v_b$. Then,

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & 0 & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \mathbf{u} .$$

P3.10 (a) From the signal flow diagram, we determine that a state-space model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} K_1 & -K_2 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \, .$$

(b) The characteristic equation is

$$\det[s\mathbf{I} - \mathbf{A}] = s^2 + (K_2 + K_1)s + 2K_1K_2 = 0 \, .$$

(c) When $K_1 = K_2 = 1$, then

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} .$$

The state transition matrix associated with $\mathbf{A}$ is

$$\mathbf{\Phi} = \mathcal{L}^{-1}\left\{[s\mathbf{I} - \mathbf{A}]^{-1}\right\} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} .$$

P3.12 (a) The phase variable representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = [40 \quad 8 \quad 0]\mathbf{x} \, .$$

(b) The canonical representation is

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \mathbf{z} + \begin{bmatrix} -0.5728 \\ 4.1307 \\ 4.5638 \end{bmatrix} r$$

$$y = [-5.2372 \quad -0.4842 \quad -0.2191]\mathbf{z}$$

(c) The state transition matrix is

$$\Phi(t) = \left[\; \Phi_1(t) \vdots \Phi_2(t) \vdots \Phi_3(t) \; \right] ,$$

where

$$\Phi_1(t) = \begin{bmatrix} e^{-6t} - 3e^{-4t} + 3e^{-2t} \\ -6e^{-6t} + 12e^{-4t} - 6e^{-2t} \\ 36e^{-6t} - 48e^{-4t} + 12e^{-2t} \end{bmatrix} \quad \Phi_2(t) = \begin{bmatrix} \frac{3}{4}e^{-6t} - 2e^{-4t} + \frac{5}{4}e^{-2t} \\ -\frac{9}{2}e^{-6t} + 8e^{-4t} - \frac{5}{2}e^{-2t} \\ 27e^{-6t} - 32e^{-4t} + 5e^{-2t} \end{bmatrix}$$

$$\Phi_3(t) = \begin{bmatrix} \frac{1}{8}e^{-6t} - \frac{1}{4}e^{-4t} + \frac{1}{8}e^{-2t} \\ -\frac{3}{4}e^{-6t} + e^{-4t} - \frac{1}{4}e^{-2t} \\ \frac{9}{2}e^{-6t} - 4e^{-4t} + \frac{1}{2}e^{-2t} \end{bmatrix} .$$

AP3.5 The differential equations describing the motion of x and θ are

$$(M+m)\ddot{x} + ML\cos\theta\ddot{\theta} - ML\sin\theta\dot{\theta}^2 = -kx$$
$$g\sin\theta + \cos\theta\ddot{x} + L\ddot{\theta} = 0$$

Assuming θ and $\dot{\theta}$ are small, it follows that

$$(M+m)\ddot{x} + ML\ddot{\theta} = -kx$$
$$\ddot{x} + L\ddot{\theta} = -g\theta$$

Define the state variables as $\mathbf{z} = \left[\; x \quad \dot{x} \quad \theta \quad \dot{\theta} \; \right]^T$. Then, the state variable model is

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m & 0 & gM/m & 0 \\ 0 & 0 & 0 & 1 \\ k/(Lm) & 0 & -g(M+m)/(Lm) & 0 \end{bmatrix} \mathbf{z}$$